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Thirteen Month Technical Progress Report

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ALTERNATIVES FOR JET ENGINE CONTROL*

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## ABSTRACT

This report documents the technical progress of researches under National Aeronautics and Space Administration Grant NSG-3048, entitled "Alternatives for Jet Engine Control", during the thirteen-month period from October 1, 1982 to October 31, 1983. NASA Technical Officer for the work was Dr. Bruce Lehtinen, at Lewis Research Center. Dr. Michael K. Sain was director of the investigation at the University of Notre Dame.

The principal new activities since the previous report have involved the initial testing of an input design method for choosing the inputs to a non-linear system so as to aid the approximation of its tensor parameters, and the beginning of order reduction studies designed to remove unnecessary monomials from tensor models. Mr. Daniel Bugajski is reporting this work, the first part of which appears in the following pages.

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# CHAPTER I

# INTRODUCTION

Inherent in the design of any control scheme is a model of the physical system or plant. The plant itself can be modelled in basically two ways. First, laws of nature can be applied to the system in question. Many of these models are time tested and still considered an adequate representation. Examples are the laws governing the mechanics of rigid bodies, and the set of equations governing the response of electric circuits. Or, second, if the system is far too complex for the relative simplicity of familiar equations, or if the governing scientific laws are too complicated to implement, system identification via excitation of the plant and measurement of the response can be performed [1]. It is this modelling technique we address here.

In the case of a linear approximation, this identification can be carried out without much computational difficulty. Unfortunately, because of system nonlinearities, in many instances the linear approximation, while adequate, leaves much room for improvement. In these cases, a model containing parameter estimates of higher degree terms (squared terms, cubic terms, and so forth) is an answer to characterizing the system nonlinearities.

Methods of calculating nonlinear models through the use of tensor algebraic ideas have been studied [2], and results have been good for academic examples [2] as well as for models of NASA's QCSE (Quiet, Clean, Shorthaul, Experimental) jet engine [3]. As we shall see, the tensor approach is invaluable in that nonlinear problems are solved using linear techniques. A challenge of this tensor method, as with all general nonlinear methods, is the problem of complexity. Obviously the size of the model will increase with the

addition of states or control inputs; however a very noticeable increase in model size accompanies any increase in the degree of approximation. Presently, typical problems using square or cubic series approximations can be handled quite capably. As the need or desire to expand to fourth degree models, and beyond, surfaces, computational constraints will need increasing attention. Upper limits on addressable storage must be considered; and most matrix software routines have a vague upper limit on dimensions beyond which calculation becomes unreliable. In lieu of computational limitations, we can call on simple intuition to cite some disadvantages in the use of large models. A first obvious observation maintains that a model containing redundant information or one retaining useless information offers no advantages over the same model with the extraneous data disregarded. Both of these faults, however, are likely to occur as model size increases. A second, less compelling reason is that large models are simply more cumbersome than smaller, compact models.

In light of the above motives, clearly a scheme to reduce full size models could be nothing but beneficial to the modelling problem. This work then opens a pathway to the identification of such reduced systems by making use of a simple idea involving the comparison of squared errors. Finally, it is worth mentioning that state reduction is not addressed here, but rather reduction via the omission of selected terms in the series approximation is considered.

A brief review of the contents of the remaining chapters is appropriate. Chapter II presents some informal mathematical background. Among the items discussed are (1) overall problem formulation, (2) input design, (3) the sin-

gular value decomposition. Chapter III briefly summarizes the software package and the logical flow of each phase of the modelling/simulation procedure. In Chapter IV the scheme for model reduction is presented and discussed. It will be shown how the reduction method fits well into the present identification technique. Sample reductions and verifications are presented in Chapter V as a few example problems are studied. Finally, Chapter VI draws some conclusions and offers some pertinent suggestions.

CHAPTER II

PROBLEM BACKGROUND

The intent of this chapter is to provide a brief presentation of some of the principles on which the work in the following chapters is founded. The purpose here is not to relay detailed theory, but the material presented will be sufficient enough to gain a proper understanding of the problem.

The model structure formation, by means of a symmetric tensor method, is discussed in the first section along with a few ideas concerning the actual calculation of the model. Following this, the next section summarizes an information theoretic approach to input design used in the parameter identification. The final section presents an overview of the singular value decomposition because of its importance in the calculation of the model.

## 2.1 MODEL STRUCTURE

In a most general form, a nonlinear ordinary differential equation in x with input u can be written

$$\dot{x} = f(x,u)$$

where x is a real n-vector and u is a real m-vector.

Now suppose a Taylor series expansion is performed around some operating point $(\bar{x},\bar{u})$, then [2]

$$\dot{\hat{x}} = f(\hat{x},\hat{u}) = f(\bar{x},\bar{u}) + \left.\frac{\partial f}{\partial \hat{x}}\right|_{(\bar{x},\bar{u})} (\hat{x}-\bar{x}) + \left.\frac{\partial f}{\partial \hat{u}}\right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \; (\hat{x}-\bar{x})^T \left. \frac{\partial^2 f_i}{\partial \hat{x}^2} \right|_{(\bar{x},\bar{u})} (\hat{x}-\bar{x})$$

$$+ \sum_{i=1}^{n} e_i \; (\hat{x}-\bar{x})^T \left. \frac{\partial^2 f_i}{\partial \hat{x} \partial \hat{u}} \right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \; (\hat{u}-\bar{u})^T \left. \frac{\partial^2 f_i}{\partial \hat{u}^2} \right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})$$

$$+ \; . \; . \; .$$

where $e_i = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \leftarrow i^{th}$ position.

If the operating point $(\bar{x},\bar{u})$ is taken as a reference, then

$$x = \hat{x} - \bar{x},$$

$$u = \hat{u} - \bar{u},$$

and if

$$\dot{\bar{x}} = f(\bar{x},\bar{u}),$$

then

$$\dot{x} = \dot{\hat{x}} - f(\bar{x},\bar{u}).$$

Substituting these into the expansion expression, we obtain

$$\dot{x} = \left. \frac{\partial f}{\partial \hat{x}} \right|_{(\bar{x},\bar{u})} x + \left. \frac{\partial f}{\partial \hat{u}} \right|_{(\bar{x},\bar{u})} u$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \; x^T \frac{\partial^2 f_i}{\partial \hat{x}^2} \Bigg|_{(\overline{x},\overline{u})} x$$

$$+ \sum_{i=1}^{n} e_i \; x^T \frac{\partial^2 f_i}{\partial \hat{x} \partial \hat{u}} \Bigg|_{(\overline{x},\overline{u})} u$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \; u^T \frac{\partial^2 f_i}{\partial \hat{u}^2} \Bigg|_{(\overline{x},\overline{u})} u$$

$$+ \ldots$$

It is at this point where tensor algebra becomes important. Specifically the bilinear symmetric tensor product, $\vee$, [4] is used to form the products of the higher degree terms. If $x \in X$ and $u \in U$, then

$$\vee : X \times X \to X \vee X$$

$$\vee : X \times U \to X \vee U$$

$$\vee : U \times U \to U \vee U$$

$$\vdots$$

We use these to write the differential equation as follows

$$\dot{x} = L_{10}x + L_{01}u + L_{20}(x \vee x) + L_{11}(x \vee u) + L_{02}(u \vee u) + \ldots$$

where the $L_{ij}$ make up the model. Note that the first and second subscripts correspond to the number of times x and u respectively are used in the symmetric tensor products associated with a given model partition $L_{ij}$. Simplifying the differential equation again

$$\dot{x} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} L_{ij} \underbrace{(x \vee x \vee x \ldots)}_{i \text{ times}} \quad \underbrace{(u \vee u \vee u \ldots)}_{j \text{ times}} .$$

Finally

$$\dot{x} = L\ z$$

where

$$L = [L_{10}\ L_{01}\ L_{20}\ L_{11}\ L_{02}\ L_{30}\ \ldots],$$

and z is a stacked vector of tensor term products:

$$z = \begin{bmatrix} x \\ \hline u \\ \hline x \vee x \\ \hline x \vee u \\ \hline u \vee u \\ \hline x \vee x \vee x \\ \hline \cdot \\ \cdot \\ \cdot \end{bmatrix} .$$

It is interesting to note that the vector z is already reduced in one sense. Because of the properties of symmetric tensor algebra, redundant cross products are eliminated. For example

$$x \vee u = u \vee x\ ,$$

$$x \vee x \vee u = x \vee u \vee x = u \vee x \vee x\ .$$

Thus, when x and u have dimension greater than one, cross products of individual elements contain redundancies and are hence eliminated. Some example instances are

$$x_1x_2 = x_2x_1,$$

$$\begin{aligned} x_1x_2x_3 &= x_1x_3x_2 \\ &= x_2x_1x_3 \\ &= x_2x_3x_1 \\ &= x_3x_1x_2 \\ &= x_3x_2x_1\ , \end{aligned}$$

$$\begin{aligned} x_1x_2u_1 &= x_1u_1x_2 \\ &= x_2x_1u_1 \\ &= x_2u_1x_1 \\ &= u_1x_1x_2 \\ &= u_1x_2x_1 . \end{aligned}$$

The existing software package includes an efficient ordering algorithm [2,3] which insures that redundant products are not calculated. It is clear that a significant amount of reduction in size, and hence, calculation is inherent is the use of symmetric tensor algebra.

We can now address the question concerning the calculation of the model, L, given a "black box" system. In our case the black box consists of a QCSEE digital simulation routine or a system of mathematical equations that represent a physical system.

Recall the equation

$$\dot{x}(t) = L\ z(t)$$

where $\dot{x}(t)$ is a vector of state derivatives and $z(t)$ is a vector of tensor terms. Because L is the unknown, obviously we must know (or estimate) the values of $\dot{x}(t)$ and $z(t)$ to identify the parameters contained in L. To get a reliable determination of how $\dot{x}$ and z change with time, the system is perturbed from an equilibrium state, $(\overline{x},\overline{u})$, to a trajectory $(\hat{x},\hat{u})$ where [5]

$$x = \hat{x} - \overline{x}$$

$$u = \hat{u} - \overline{u}.$$

The values of $x(t)$ and $u(t)$ and estimates of $\dot{x}(t)$ are then sampled and stored. From $x(t)$ and $u(t)$, $z(t)$ is formed. The problem now appears:

$$\begin{bmatrix} \dot{x}_1(t_1) & \dot{x}_1(t_2) & \cdots & \dot{x}_1(t_h) \\ \dot{x}_2(t_1) & \dot{x}_2(t_2) & \cdots & \dot{x}_2(t_h) \\ \vdots & \vdots & & \vdots \\ \dot{x}_n(t_1) & \dot{x}_n(t_2) & \cdots & \dot{x}_n(t_h) \end{bmatrix} = L \begin{bmatrix} z_1(t_1) & z_1(t_2) & \cdots & z_1(t_h) \\ z_2(t_1) & z_2(t_2) & \cdots & z_2(t_h) \\ \vdots & \vdots & & \vdots \\ z_p(t_1) & z_p(t_2) & \cdots & z_p(t_h) \end{bmatrix}$$

where h is the number of samples and p is the number of terms in the nonlinear approximation; p is dependent on the number of states, n, the number of inputs, m, and the degree of approximation. We shall write this as $\dot{X} = LZ$.

Fortunately, a very good routine for solving a problem of this form exists in the SPEAKEASY library of the IBM 370/3033. Using a singular value decomposition, discussed later, of Z, a least squares problem is solved [6] and L is calculated. The model then is ready for verification by means of digital simulation and comparison with true system solutions.

## 2.2 INPUT DESIGN

From the preceding section, it is clear that the model L is dependent on the excitation, $(x(0), u(t))$. Improper choice of perturbation might translate to an insufficient excitation of the nonlinearities, instability, or singularity. A careful method for choosing inputs could decrease the likelihood of encountering these defects. With this, the chance of identifying a "good" model, one with strong tracking ability and an acceptable region of stability, is then increased. In addition to better models, the capability of an input selection routine eliminates the need for an exhaustive search of input parameters.

For the purpose of this work, the theoretical detail of an input optimization derivation is not required. It is the intent here merely to present the basic members in the cost function calculation. For a theoretical and mathematical treatment, the reader is directed to [7]. The basic idea for the optimization comes from [8] and leads to the introduction of Fisher's Information Matrix, M. It is desired to get, in some sense, the minimum of a measure of $M^{-1}$. The formulation of Fisher's Information Matrix has its roots in probability, and many probabilistic rules and simplifications are used to express each element of M as

$$M_{ij} = \sum_{k=0}^{N} \left[ \frac{\partial}{\partial \theta_i} y(k) \right]^T R^{-1} \left[ \frac{\partial}{\partial \theta_j} y(k) \right],$$

where

$$i = 1,2,\ldots np,$$

$$j = 1,2,\ldots np,$$

and

N is the number of observation points,
y(k) is the model output sequence,
$\theta$ is a vector of parameters to be identified,
R is the covariance matrix of the measurement of y(k).

Now for our problem let

$$y = L\ z,$$

or for the discrete case

$$y(k) = L\ z(k).$$

Recall that $\theta$ is a vector of parameters to be identified, that is, each element of L. The partial derivatives can then be found. If

$$\underbrace{y}_{n\times 1} = \underbrace{L}_{n\times p}\ \underbrace{z}_{p\times 1},$$

then the partial derivative of $y = Lz$ with respect to some element of L, say $\ell_{ij}$, is an nxl matrix with zeros in all but the i-th position where the entry is simply the j-th entry of z, $z_j$. Now let

$$R^{-1} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix};$$

upon the calculation of each $M_{ij}$ and the building of M, the following observation is made [7]

$$M = R^{-1} \otimes \overline{M}$$

where $\overline{M}$ is symmetric and its upper triangular portion is shown in Figure 2.1.

Now that we have determined that $\overline{M}$ and hence M are calculable, their role in the cost function is examined. The objective function chosen is

$$\min \operatorname{tr}(M^{-1}).$$

We use a property of the Kronecker product [9] to obtain

$$M^{-1} = R \otimes \overline{M}^{-1}.$$

Another property allows

$$\operatorname{tr}(M^{-1}) = \operatorname{tr}(R)\operatorname{tr}(\overline{M}^{-1}).$$

But if $\overline{M}$ is an invertible matrix,

$$\operatorname{tr}(\overline{M}^{-1}) = \sum_{i=1}^{p} \frac{1}{\lambda_i}$$

where the $\lambda_i$ are the eigenvalues of $\overline{M}$. Finally

$$\operatorname{tr}(M^{-1}) = \operatorname{tr}(R)\left( \sum_{i=1}^{p} \frac{1}{\lambda_i} \right).$$

It is worthwhile to note that, following from the above discussion, there are essentially two ways to generate the objective value; the eigenvalues of $\overline{M}$ or the inverse of M must be calculated. In the case of real, symmetric matrices, these are similar in software requirements. Because we only need the trace, and not the entire inverse, we choose the eigenvalue approach.

$$
\overline{M} = \left[\begin{array}{c|c|c|c|c|c}
\sum x \otimes x^T & \sum x \otimes u^T & \sum x \otimes (x \vee x)^T & \sum x \otimes (x \vee u)^T & \sum x \otimes (u \vee u)^T & \cdots \\
\hline
 & \sum u \otimes u^T & \sum u \otimes (x \vee x)^T & \sum u \otimes (x \vee u)^T & \sum u \otimes (u \vee u)^T & \cdots \\
\hline
 & & \sum (x \vee x) \otimes (x \vee x)^T & \sum (x \vee x) \otimes (x \vee u)^T & \sum (x \vee x) \otimes (u \vee u)^T & \cdots \\
\hline
 & & & \sum (x \vee u) \otimes (x \vee u)^T & \sum (x \vee u) \otimes (u \vee u)^T & \cdots \\
\hline
 & & & & \sum (u \vee u) \otimes (u \vee u)^T & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]
$$

Figure 2.1. Upper Triangular Portion of the Matrix $\overline{M}$

This distinction becomes more important with increasing model size, which grows factorially with n, m, and degree of approximation. Despite the fact that matrix inversion is less acceptable, the current software makes provisions for both methods, for purposes of generality.

## 2.3 SINGULAR VALUE DECOMPOSITION

The discussion now turns to a brief presentation of the singular value decomposition (SVD). In this research effort, the SVD serves the important purpose of calculation of the generalized inverse of a matrix when the model is to be determined. The reader is directed to [10,11,12] for more formal and theoretically based discussions concerning the SVD and its computational implications and interpretations.

We first present the defining theorem of the SVD. If A is nxm with rank $\rho$, then there exist orthonormal matrices U (nxn) and V (mxm) such that

$$A = USV^T$$

where

T indicates transposition

and

S (nxm) is a "diagonal" matrix with diagonal elements $s_i$,

$$s_i > 0 \quad i = 1,2,\ldots\rho$$

$$s_i = 0 \quad i = (\rho+1)\ldots\min(n,m) .$$

The columns of U are the left singular vectors and are the orthonormal eigenvectors of AA*. The columns of V are the right singular vectors and are the orthonormal eigenvectors of A*A. As expected the diagonal elements of S are the singular values. The non-zero elements of $\{s_i\}$ are the positive square roots of the eigenvalues of A*A (or AA*). This may seem to suggest a simple

way to calculate the singular values. This is not true, however, when finite precision arithmetic is involved. Because of the usually inexact representations due to truncation or rounding of numbers in a digital computer, the calculation of the eigenvalues of A*A can often lead to incorrect results concerning matrix rank [11,12].

Fortunately, SVD algorithms do exist that are computationally sound and not hindered by the above fault. Those that are generally considered among the best are the versions developed by Argonne National Laboratories. These include the SVD subroutine used by the SIMEQUAT algorithm of the SPEAKEASY library, IBM system 370/3033. The SIMEQUAT routine solves for the minimal solution x to the equation Ax=b. It does this by decomposing A and then finding the generalized inverse, $A^I$, (or $A^{-1}$ if A is square) of A. Given the generalized inverse, the problem becomes trivial. If

$$A = USV^T$$

then

$$A^I = VS^IU^T$$

and

$$x = A^Ib$$

becomes

$$x = VS^IU^Tb.$$

Despite the existence of other methods for calculating the generalized inverse, $A^I$, the SVD remains the best. This is so because in problems where matrix rank is involved, as it is for matrix inversion, the SVD is the most reliable method of rank determination [11,12].

Finally, the concept of condition number is introduced. Very simply the condition number, k, is the ratio of the largest singular value to the smal-

lest non-zero singular value. That is

$$k = \frac{\max s_i}{\min s_i}.$$

There is a lot of abstract theory involving the interpretation of the condition number, but for the purposes herein, we will regard the condition number as a measure of "nearness to singularity".

## 2.4 REMARKS

This chapter presents the major ideas involved in the present modelling scheme. A sound, qualitative understanding of section one is most important, since throughout the remainder of this thesis, references to these ideas will be made. In section two, a summary derivation of the input design is presented. Here we are concerned not with mathematical rigor but rather with the idea that an input can be designed at all. We shall see the importance of this design when the examples are discussed.

# CHAPTER III

## SOFTWARE OVERVIEW

The intent of this chapter is simply to present the logical flow of the software package. This includes both the modelling and the simulation segments. Most of the CATNAP [3] software still exists, although in a more flexible form. In addition, software has been added to provide for the input optimization as well as for the model reduction test as presented in Chapter IV. The interactive modelling and simulation packages make use of two computers. They are the IBM 370/3033 and the DEC PDP 11/44.

### 3.1 MODELLING

The modelling segment of the present software package ties together each of the three ideas found in Chapter II. The IBM 370/3033 computer is used here in light of its computational power and its extensive support software libraries. The routine SUPRVIZE is an IBM command language (CLIST) program which governs the entire modelling segment. It is in SUPRVIZE that necessary libraries are made accessible and control is passed from loader routine to identification routine.

The first major duty of SUPRVIZE is compilation and execution of the proper loader routine. The loader routine (usually written in extended precision FORTRAN) has the chore of set up for identification and/or optimization. It is first determined whether a reduced model is to be identified. If so, a set of column numbers corresponding to the column numbers of the full model is read from the data set REDUCE. The products that correspond to these column numbers are omitted from further calculation. Then if an optimization is desired, the proper arrays are initialized and the minimization routine is

called. The minimization routine is supplied by IMSL (International Mathematical and Statistical Libraries) available on the IBM 370/3033. The IMSL routine requires a user supplied subroutine which calculates the cost function. In our case the cost function is computed using the eigenvalues of Fisher's Information Matrix as discussed in Section 2.2. Following the determination of inputs (whether by optimization or by user choice), the loader routine excites the system and forms the tensor term and state derivative matrices via sampling. These two matrices are loaded into the data set TEMPFILE for later use.

After termination of the loader execution, the next duty of SUPRVIZE is to invoke the high level language SPEAKEASY [6]. The routines written in SPEAKEASY have two tasks: model calculation, and reduction test, if desired. As we have mentioned in Section 2.1, the model is calculated using a least squares approximation and singular value decomposition in the SIMEQUAT function of the SPEAKEASY library. Following output of the model parameters, the model is stored in data set MODEL. Now the reduction test can be performed. After completion of the test, if model reduction/re-optimization is desired, a set of column numbers is written to the data set REDUCE. SPEAKEASY is then exited and control is passed back to SUPRVIZE.

The final tasks of SUPRVIZE are simple. If the model is to be kept, the user is prompted for a model name and storage area (one of two partioned data sets). The model then is stored in the desired partioned data set with the given name. If another model is to be identified, the procedure restarts. Otherwise, unnecessary data sets are deleted and modelling is complete. See Figure 3.1 for a flow diagram of the entire modelling scheme.

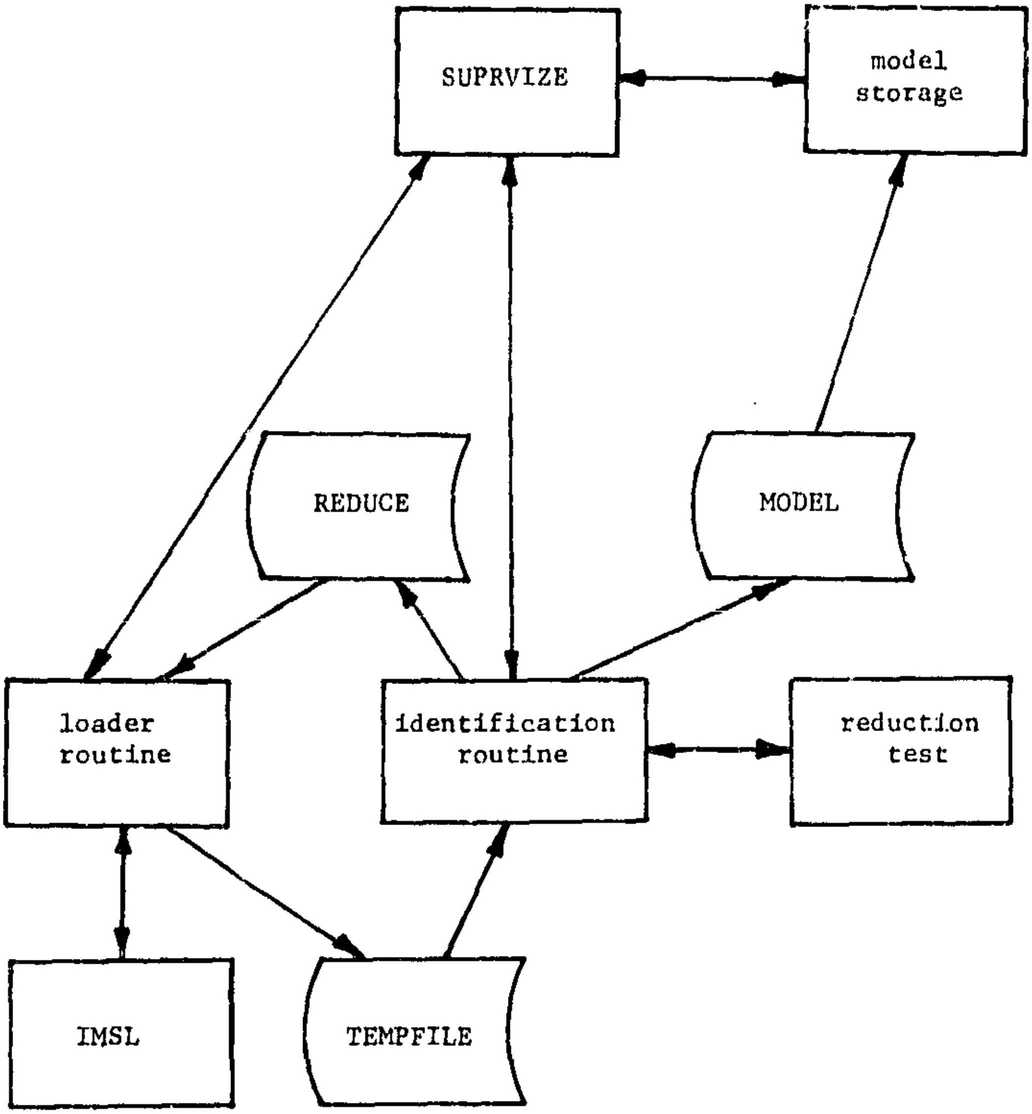



Figure 3.1. Logical Structure of Modelling Scheme

## 3.2 SIMULATION

Simulation of the nonlinear models can take place on either the IBM 370/3033 or the DEC PDP 11/44. For bulk tabular verification, the IBM 370/3033 is generally preferred due to its computational speed. Plot capability is handled with SAS (Statistical Analysis Software). For plot comparison, though, it is preferred to use the DEC PDP 11/44. While computationally slower than the IBM, the DEC machine has high speed CRT plot capability with a hard copy unit for instant quality plots.

Simulation on the IBM 370/3033 is governed by the CLIST program COMPARE. COMPARE has several duties, the first of which is compilation of the simulation routine. The simulation routine (usually written in single precision FORTRAN) has the chore of solving the true system and integrating one or two models for a specified time interval. Furthermore, the simulation routine stores the data in the case that plotting is desired, and performs an error analysis between the models (if two models are simulated simultaneously). The error criteria is a simple ratio of mean square errors of the two models for each state.

After the simulation routine has been compiled, control passes back to COMPARE where the set up for simulation takes place. If a bulk tabular simulation is desired, COMPARE compiles and runs a short program that builds any specified table and stores it in data set SIMPNT. Upon exit from COMPARE, SIMPNT can be copied to a partitioned data set, TABLES, which acts as a library of simulation tables. If the desired simulation table is already built, COMPARE will ask for its name along with the names of the desired models. Following this, execution of the simulation routine is started. If the simulation is at a single point, the user is prompted for the input parameters.

Otherwise, the table is read for the input parameters of each simulation. After all simulations have been completed, control is passed back to COMPARE. If plotting is desired, control is passed to the SAS plotting routine. Otherwise the option to do another simulation is made. If simulation is complete, COMPARE is exited. See Figure 3.2 for the logical flow of simulation on the IBM 370/3033.

For simulation on the PDP 11/44, the models must first be transferred via magnetic tape from the IBM 370/3033 . Once the model is on the DEC machine, an updated version of the CATNAP simulation routine is used. See Section 3.3 of [3] for an explanation of this simulation structure. Here we list only major changes to the old CATNAP software. They are

1) comparison ability between any two models and the true solution (as opposed to strictly true solutions, linear model and some nonlinear model),
2) instead of a separate Versatec hard copy routine, a hard copy unit which transfers the CRT image to paper is used,
3) the error criteria is now the ratio of mean square errors,
4) ability to blow up a portion of any plot for closer inspection of model trajectories.

In the writing of the current software, every effort was made to keep the programs general and flexible. See [15] for current IBM 370/3033 software listings. See also [16] for current listings of PDP 11/44 software.

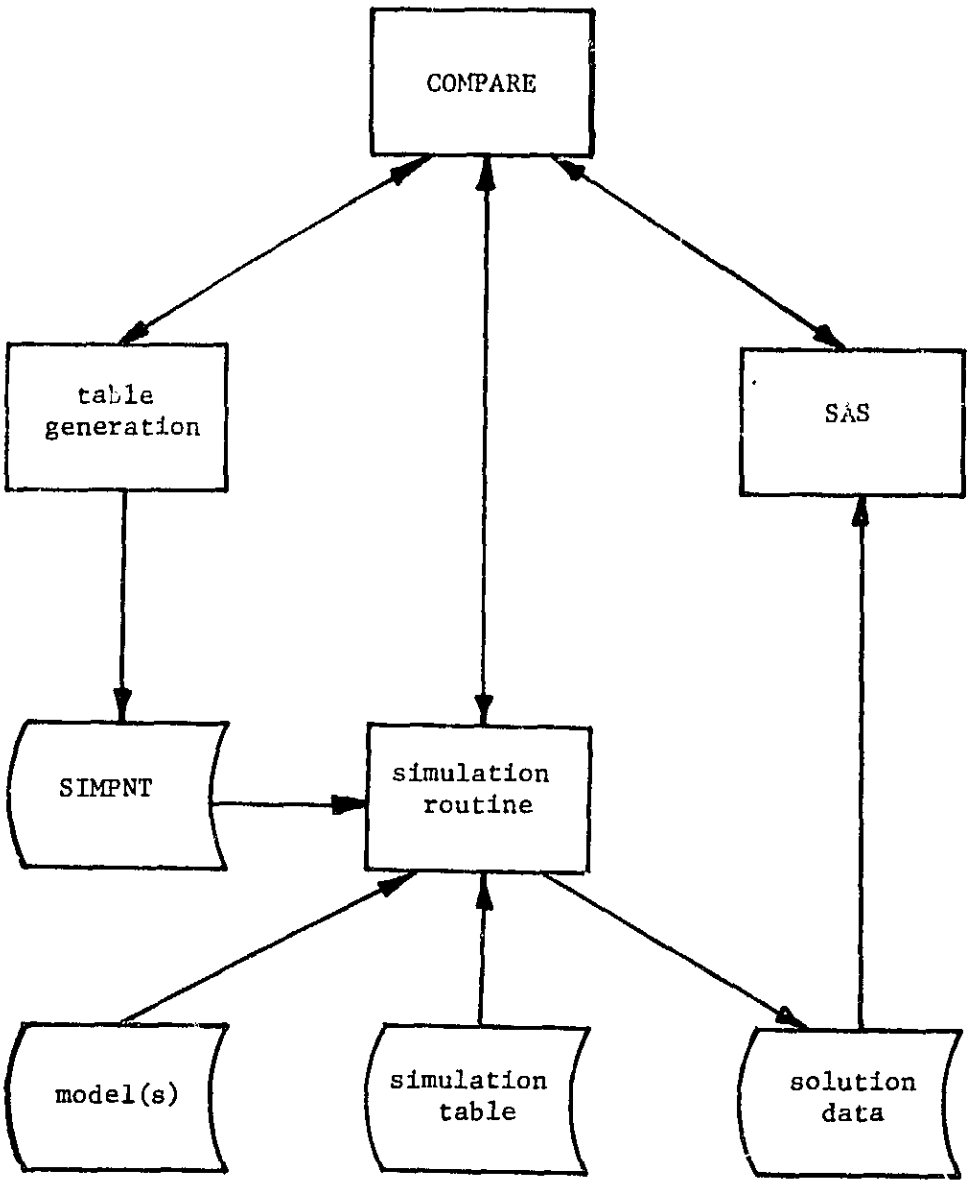



Figure 3.2. Logical Struction of Simulation on IBM 370/3033

# CHAPTER IV

# MODEL REDUCTION

In Chapter II some mathematical ideas were set forth that are at the center of the entire problem of nonlinear modelling via tensor parameterization. It is hoped in any nonlinear modelling exercise that the addition of higher degree terms will benefit the model in that the system nonlinearities will be more easily characterized by these higher degree terms or combinations thereof. But these models with the obvious advantage of higher degree lead us to a challenge. This challenge is the dimension or size of the model, p, and the arrays associated with its computation. Figure 4.1 shows how the size of model increases with the increase of the three determining factors, number of states, number of controls and degree. As the model gets larger, several issues may need to be examined, such as addressable storage limits, limits on computational reliability, and time of computation. Clearly a way of making the models easier to handle is very desirable. In short, if modelling is to proceed to higher degree terms, the models must have unnecessary monomials eliminated. It is to this end that we devote the remainder of this chapter to the introduction of one such size reduction method.

## 4.1 REDUCTION TECHNIQUE

As mentioned in past chapters, the model is calculated using a least squares approximation given times series state derivative data and time series tensor product data. Because this calculation uses a least squares approximation, and a truncation of the series approximation there will be an inherent error in the model. We can characterize the error, E, as

$$E = \dot{X} - L\ Z.$$

| number of states | number of controls | degree of approximation | model size |
|---|---|---|---|
| 2 | 2 | 1 | 2 x 4 |
| 2 | 2 | 2 | 2 x 14 |
| 2 | 2 | 3 | 2 x 34 |
| 2 | 2 | 4 | 2 x 69 |
| 2 | 3 | 1 | 2 x 5 |
| 2 | 3 | 2 | 2 x 20 |
| 2 | 3 | 3 | 2 x 55 |
| 2 | 3 | 4 | 2 x 125 |
| 2 | 4 | 1 | 2 x 6 |
| 2 | 4 | 2 | 2 x 27 |
| 2 | [illegible] | 3 | 2 x 83 |
| 2 | [illegible] | 4 | 2 x 209 |
| 3 | 2 | 1 | 3 x 5 |
| 3 | 2 | 2 | 3 x 20 |
| 3 | 2 | 3 | 3 x 55 |
| 3 | 2 | 4 | 3 x 125 |
| 3 | 3 | 1 | 3 x 6 |
| 3 | 3 | 2 | 3 x 27 |
| 3 | 3 | 3 | 3 x 83 |
| 3 | 3 | 4 | 3 x 209 |
| 3 | 4 | 1 | 3 x 7 |
| 3 | 4 | 2 | 3 x 35 |
| 3 | 4 | 3 | 3 x 119 |
| 3 | 4 | 4 | 3 x 315 |
| 4 | 2 | 1 | 4 x 6 |
| 4 | 2 | 2 | 4 x 27 |
| 4 | 2 | 3 | 4 x 83 |
| 4 | 2 | 4 | 4 x 209 |
| 4 | 3 | 1 | 4 x 7 |
| 4 | 3 | 2 | 4 x 35 |
| 4 | 3 | 3 | 4 x 119 |
| 4 | 3 | 4 | 4 x 315 |
| 4 | 4 | 1 | 4 x 8 |
| 4 | 4 | 2 | 4 x 44 |
| 4 | 4 | 3 | 4 x 164 |
| 4 | 4 | 4 | 4 x 474 |

Figure 4.1. Variation of Model Sizes with Various Parameters

Because $\dot{X}$ and Z are time series data, E will become a time series matrix of state derivative errors. This simple calculation will become the base for the reduction method.

The idea of using an error matrix analysis is a modified approach from a treatment of nonlinear model reduction given by A.A. Desrochers [13]. Desrochers poses the problem with only a linear control input and in discrete time

$$x(k+1) = A\ F(k) + B\ u(k)$$

where x is a state vector,
u is an input,
A and B constitute the model,
F is q-vector of states and state combinations, that is, $F(k) = \tilde{F}(x(k))$.

To determine which columns[1] of A are most dominant, a modified state error vector $\tilde{e}$ is formed

$$\tilde{e}_j(k) = x(k+1) - [A(,j)B] \left[\begin{array}{c} F(j,k) \\ \hline u(k) \end{array}\right] \quad j = 1,2,\ldots,q.$$

Note that the error is modified in the sense that it is formed using only one term at a time in addition to the control input. The state error vectors are then squared and summed for all h time points

$$\sum_{k=0}^{h-1} \tilde{e}_j^T(k)\ \tilde{e}_j(k).$$

Based on this squared error number, the most important terms can be kept and the others discarded.

Now in our more general problem, we have nonlinear input terms in addition to the linear terms, and we choose to test the effect of all terms in-

[1] Henceforth, given a matrix A, we denote the ij-th element as A(i,j), the i-th column of A as A(,i), and the i-th row as A(i,).

cluding those corresponding to the linear control inputs. (Recall Desrochers decided to keep all linear controls.) Our test error matrix $E_j$ is

$$E_j = \dot{X} - L_j Z_j \,, \qquad j = 1,2,\ldots,p$$

where $E_j$ is the time series error matrix with the j-th term removed,
$p$ is the number of terms in the approximation,
$L_j$ is the model with the j-th column removed,
$Z_j$ is the time series tensor term data with the j-th row removed.

Note that now, in addition to the error incurred by the averaging of the least squares algorithm, there is error due the loss of the j-th term. We see that each column of $E_j$ is the error vector at a certain time. Then we can use the same squared error calculation,

$$\sum_{k=0}^{h-1} e_j^T(k)\, e_j(k) \qquad j = 1,2,\ldots,p$$

where $E_j = [e_j(0) \mid e_j(1) \mid \ldots \mid e_j(h-1)]$, to determine the effect of a certain term on the model. The influence of a term on the model is measured by checking the difference of the squared error of the full model and the squared error of the model shortened by that term. As an aside, it is interesting to note a philosophical difference between the problem formulation of Desrochers and this problem formulation. Desrochers chose to pick a significant term by including only that term in the error test. On the other hand, our reduction test checks a term's significance when that term is removed.

A simple example here will illustrate the calculations. Suppose we have a two state, one control system which we sample for four time points and calculate a linear approximation. Let

$$\dot{X} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}.$$

First we calculate the state error matrix E of the full model, $E = \dot{X} - LZ$,

$$E = \begin{bmatrix} -7 & -12 & -5 & -5 \\ -5 & -9 & 3 & -9 \end{bmatrix}.$$

Now calculate the squared error for the full model, $\Psi$,

$$\Psi = \sum_{k=0}^{3} E^T(,k)\ E(,k) = [-7 \quad -5] \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$+ [-12 \quad -9] \begin{bmatrix} -12 \\ -9 \end{bmatrix}$$

$$+ [-5 \quad 3] \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$+ [-5 \quad -9] \begin{bmatrix} -5 \\ -9 \end{bmatrix}$$

$$= 439.$$

Following this calculation, we start checking the errors of the shortened models. We want first to calculate $E_1$ which is the state error matrix with the first term of the model, $x_1$, and its corresponding data removed. According to the method, the first row of z is removed, the first column of L is removed, and the error calculations are repeated:

$$E_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -11 & -3 & 1 \\ -5 & -7 & 7 & -3 \end{bmatrix}.$$

$$\Psi_1 = \sum_{k=0}^{3} E_1^T(,k)\ E_1(,k) = [-7 \quad -5] \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$= [-11 \quad -7] \begin{bmatrix} -11 \\ -7 \end{bmatrix}$$

$$+ [-3 \quad 7] \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$= 312.$$

Then the relative error due to term $x_1$ is $\Psi - \Psi_1 = 312-439 = -127$.

Now we repeat the calculations for the removal of the second term in the model, $x_2$. Here

$$E_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}.$$
$$= \begin{bmatrix} -5 & -8 & 1 & -2 \\ -6 & -11 & 0 & -9 \end{bmatrix}.$$

The squared error number, $\Psi_2$,

$$\Psi_2 = \sum_{k=0}^{3} E_2^T(,k)\ E_2(,k) = 332,$$

and the relative error is $332-439 = -107$.

Finally the last term, corresponding to u, is checked. Thus

$$E_3 = \begin{bmatrix} -1 & -3 & -5 & 1 \\ 3 & 3 & 3 & -5 \end{bmatrix},$$

$$\Psi_3 = \sum_{k=0}^{3} E_3^T(,k)\ E_3(,k) = 88,$$

and the relative error is $88-439 = -351$.

In most cases, examination and comparison of the computed relative errors will give a fair indication of which terms in the model are the most influential. Sometimes this is not the case and we must extend this squared error concept to look at more data. The extension is a simple and informative one.

Recall that each error matrix, E and the $E_j$'s, is a time series matrix representation of the errors in the states. For example, in the previous

example the time series error in the first state of the full model is

$$E(1,) = [-7 \quad -12 \quad -5 \quad -5],$$

the first row of E. Now use the squared error calculation for each individual state to get

$$\psi_i = \sum_{k=0}^{h-1} E(i,k)\, E(i,k) \qquad i = 1,2,\ldots,n.$$

This says that the error in the i-th state, $\psi_i$, is the sum of the squares of each of the entries of the i-th row of E. Furthermore note that the error in the entire system, $\Psi$, is the sum of the $\psi_i$'s. That is

$$\Psi = \sum_{i=1}^{n} \psi_i,$$

and for the reduced cases

$$\Psi_\ell = \sum_{i=1}^{n} \psi_{\ell i},$$

where $\ell$ is the number of the term removed from the model. With these additional calculations, now, we can compare not only the entire system errors (the $\Psi - \Psi_\ell$), but also the errors in the states $\psi_i - \psi_{\ell i}$.

To illustrate these calculations, we return to the example. We begin with the full model. The error in the first state is

$$\begin{aligned}\psi_1 &= \sum_{k=0}^{3} E(1,k)E(1,k) \\ &= (-7)^2 + (-12)^2 + (-5)^2 + (-5)^2 \\ &= 243,\end{aligned}$$

and the error in the second state is

$$\begin{aligned}\psi_2 &= \sum_{k=0}^{3} E(2,k)E(2,k) \\ &= (-5)^2 + (-9)^2 + (3)^2 + (-9)^2\end{aligned}$$

$$= 196.$$

Thus

$$\Psi = \psi_1 + \psi_2 = 243 + 196 = 439.$$

After removing the first term in the model, the error in the first state is

$$\psi_{11} = \sum_{k=0}^{3} E_1(1,k)E_1(1,k)$$

$$= (-7)^2 + (-11)^2 + (-3)^2 + (1)^2$$

$$= 180,$$

and the error in the second state is

$$\psi_{12} = \sum_{k=0}^{3} E_1(2,k)E_1(2,k)$$

$$= (-5)^2 + (-7)^2 + (7)^2 + (-3)^2$$

$$= 132.$$

So

$$\Psi_1 = \psi_{11} + \psi_{12} = 180 + 132 = 312.$$

Now, as we calculated before, the change in the overall error is $\Psi_1 - \Psi = -127$. In addition to this we check the change in the state errors. That is,

$$\psi_{11} - \psi_1 = 180 - 243 = -63$$

and

$$\psi_{12} - \psi_2 = 132 - 196 = -64.$$

Now, repeat the calculations when only the second term of the model is removed.

$$\psi_{21} = \sum_{k=0}^{3} E_2(1,k)E_2(1,k)$$

$$= (-5)^2 + (-8)^2 + (1)^2 + (-2)^2$$

$$= 94$$

$$\psi_{22} = \sum_{k=0}^{3} E_2(2,k)E_2(2,k)$$

$$= (-6)^2 + (-11)^2 + (0)^2 + (-9)^2$$

$$= 238.$$

Then
$$\Psi_2 = \psi_{21} + \psi_{22} = 94 + 238 = 332,$$

$$\Psi_2 - \Psi = 332 - 439 = -107,$$

and

$$\psi_{21} - \psi_1 = 94 - 243 = -149$$

$$\psi_{22} - \psi_2 = 238 - 196 = 42.$$

Finally, the third term is removed and the calculations are performed. We get

$$\psi_{31} = 36$$

$$\psi_{32} = 52$$

and $\Psi_3 = 88$, so $\Psi_3 - \Psi = -351$.

Also
$$\psi_{31} - \psi_1 = -207$$

$$\psi_{32} - \psi_2 = -144,$$

which finishes the computation.

Arranging this data in a table will make it readable and usable. See Figure 4.2 for the reduction data of the sample problem. Naturally, as the model increases in size, the testing will increase to include all terms in the model approximation. These, then, are the simple calculations concerning the reduction method. When a reduction is done in the following chapter, an explanation of how to use these numbers will be presented.

Assume for now that the above test has indicated a set of terms are likely candidates for removal. There are several steps that can now be taken.

Full model system error: 439

Full model state errors: 243 196

Term: $x_1$

Change in system error: -127

Change in state errors: -63 -64

Term: $x_2$

Change in system error: -107

Change in state errors: -149 42

Term: $u_1$

Change in system error: -351

Change in state errors: -207 -144

Figure 4.2 Reduction Data for Sample Exercise

One option is to remove those terms from the model and retain the remainder of the terms thus producing the reduced model. This, although simple, is not a logical choice. For when we omit terms in a model, we are overlooking some dynamics. Regardless of their significance, these lost dynamics may be modelled adequately using the remaining terms, if they are given a chance. Hence, it is advantageous to regenerate the derivative and tensor data followed by a re-calculation of the now reduced model. But, by removing terms, we have changed the model structure and, intuitively, the system will need to be excited differently. Therefore, it would be a further advantage if a new input is designed.

The decision to re-optimize after terms are omitted seems sound, but it is by no means trivial. It was argued that a new input design is the sensible next step after the decision is made as to which terms are kept. But if the input changes (due to the optimization), the full model associated with this new excitation may contain influential terms at those places we have already made unavailable by reduction. Because of this difficulty, an assumption is made. It is assumed that a relatively insignificant term will remain relatively insignificant if the change in input excitation is not extreme. To ensure that the change in excitation is not too great, terms are not discarded in large groups. This is so, because it is hypothesized that a large change in model structure (that is, many terms removed) will cause the optimum input to move farther away from the present input parameters. Furthermore, the input optimization routine converges to a local minimum which will usually ensure the parameters do not move too much. To summarize, problems in reduction may be encountered if there is significant change between the input parameters of the model to be reduced and the input parameters of the reduced model.

As a result of the previous discussion, we have made a minor rule for reduction. It claims that terms to be omitted should not be discarded in large groups. There is another rule implicit in the preceding paragraph which is worth stating. The model to be reduced must have been identified at an input parameter set which is locally optimum. If it is not, optimization for the reduced model may change the input parameters significantly, and we have already argued that, in general, this is not desired.

It is worthwhile to pause here to make some observations. First, by way of reasoning, a "multiple pass" reduction method has evolved. That is, an optimum input is chosen, the model is calculated, and the least significant terms in the model are determined. These terms are removed from further calculation, and an optimum input is re-chosen, the model is re-calculated, and so on. Obviously this process has a limit, and as the model becomes smaller, more care must be taken when choosing terms to be omitted. In some cases, the final pass is obvious. In these instances, the reduction test will indicate that all remaining terms are relatively significant. In other cases, the final pass is not so clear. Usually after a few passes, the model is adequately shortened, and the reduction test may indicate only a single term or two to be insignificant. It is in these cases that care must be taken when omitting terms. This is intuitively sensible since, as a model is made smaller, the dynamics in a single term are harder pressed to appear in the remaining terms. Because of this, model performance may suffer. It is in such cases that after a model has been reduced, some sample simulations should be run to check if model performance is still acceptable. If it is not, then the previous step was the final one. Otherwise reduction can continue. A second observation is that in requiring only a small change in input parameter design

from step to step in the reduction process, we have taken advantage of the fact that the IMSL optimization routine yields a local minimum. While it would be more advantageous to have a global minimization routine if full model identification was being done, in the case of iteratively reducing models, the local optimization is actually more desirable.

Returning to the reduction method, we can use the rules from the previous arguments to state the scheme in full:

a) identify a model at an input which is locally optimum,
b) determine a set of terms which seem to be the least significant.
c) remove the terms in the set and re-optimize,
d) identify reduced model,
e) model simulation to test performance,
f) if reduction data indicates more insignificant terms, go to step b), otherwise stop.

This simple method works quite well as the examples in the following chapter will show.

## 4.2 REMARKS

This chapter summarizes the present reduction method, including demonstration of the calculations and a step by step outline of the technique. Difficulties arising from the combination of input design and term removal are discussed and a suitable compromise is obtained by the application of some basic rules. The following chapter contains concrete examples where the qualitative ideas of this chapter become clearer.

# CHAPTER V

# EXAMPLE REDUCTIONS

It is in this chapter that a few examples are studied to illustrate the power and usefulness of the model reduction method presented in the previous chapter. The first two systems were initially investigated in [2]. The first system is a two state, two control input set of nonlinear differential equations. The second system is a three state, three control set of strictly polynomic nonlinear differential equations. The final system is a three state, two input example. The first system will be modelled and reduced for second and third degree approximations. The other two examples will demonstrate second degree reductions.

The true test of a model's validity is borne out in simulation. Therefore much of the chapter is comprised of simulation data tables and plots. With this in mind, we establish two general criteria for judging a reduced model. They are:

a) the reduced model should approximate the full model of like degree of approximation,

b) the reduced model should outperform the full model of less degree.

Both of these are sensible goals and the examples herein do well with respect to them. Finally, it is worth restating a weak assumption necessary for model reduction. This is that the model should be an acceptable representation of the system. This is fairly obvious: since the motivation of a reduced model is to approximate the full model, we would hope the full model accurately describes the true system.

## 5.1 TWO STATE SYSTEM

The first system is given by the following equations

$$\dot{x}_1 = u_2\cosh(x_1x_2) - e^{2u_1}\sinh(2x_1) - 3\sinh(x_2),$$
$$\dot{x}_2 = e^{u_1u_2}\sinh(x_1) - e^{u_1}u_1\cosh(x_1^2) + \sinh(x_2).$$

These equations were studied in depth to demonstrate the power of the input optimization in identifying models that improve in quality as the degree of approximation increased. This study appears in Appendix A. It is from this report that "good" second and third degree models are taken as starting points for the reduction.

### 5.1.1 DEGREE TWO APPROXIMATION

For this system, the point of expansion is the origin because it is a stable equilibrium point. In the case of a second degree expansion, 14 model parameters corresponding to the tensor products

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1x_2,\ x_2x_2,$$
$$x_1u_1,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_1,$$
$$u_1u_2,\ u_2u_2$$

must be identified.

Now, to begin the reduction, a good full model that is locally optimum is necessary. The first appendix gives us such a model. That particular second degree model was identified using an initial state perturbation vector of

$$x_0 = (0.005,\ -0.005).$$

In addition the excitation for each input was taken to be a sum of two sinusoids:

$$u_1(t) = 0.025\sin(2\pi t\cdot\phi_1) + 0.025\sin(2\pi t\cdot 2\phi_1),$$
$$u_2(t) = 0.025\sin(2\pi t\cdot\phi_2) + 0.025\sin(2\pi t\cdot 2\phi_2).$$

In this case $\phi = (\phi_1, \phi_2)$ was determined by optimization to be (0.7418, 0.9022) given the starting values (0.75, 1.0). The system was run for 4 seconds and 100 samples were taken at evenly spaced intervals.

Given this locally optimum model, the reduction test can be performed. See Figure 5.1.1 for the full model and its reduction test. The decision as to which terms should be omitted (or equivalently which terms should be kept) must now be made. Upon inspection of the reduction test, it should be noted that with the removal of any term, the change in the squared error number may increase or decrease significantly, or stay about the same. In this particular application, we choose to ignore the sign of the square error numbers and look upon their magnitudes as quantifiers of the effect of a term on the model. This is reasonable since removal of a term with a large change in squared error (either an increase or decrease) will most likely cause a large change in model performance. This, then, directs our attention to the terms with small changes in their errors. These terms are in some sense the least significant, so it is hoped that removal of them will have the least effect on model performance.

As a conservative first attempt, it is decided that error changes (both system and state errors) with magnitudes less than about 4 will be omitted. The set of these terms is

$$x_1x_1,\ x_1x_2,\ x_2x_2,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_2.$$

Equivalently, the terms kept are

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1u_1,\ u_1u_1.$$

Note that all linear terms were kept. In general, saving the linear terms is desired, since they hold the good local behavior.

Now, an input optimization is performed using only the six significant terms given by the reduction test The system is again perturbed with initial conditions

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0050      -0.0050

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100     0.0400     0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2           2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.24999999999999996E-01  0.24999999999999996E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.7418       0.9022

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N


ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  14 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:         0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N




TSO SPEAKEASY III PI+  7:16 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
```

Figure 5.1.1a  Full Second Degree Model and Reduction Test (First Pass)

ORIGINAL PAGE IS
OF POOR QUALITY

```
 -2.0009 -2.9978
  .9998   .99982

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50055+.86378i -.50055-.86378i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.7287E-4  .99934
 -1.0003    -3.5733E-5


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  .0019401  -.080981    .034007
  .0052751   9.3478E-4 -3.8931E-4


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0686    -.0062571 -.030538  -.024848
 -.018969   -.011543  -.01223   -.011688


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.033944   -6.0501E-4  .0049369
 -.99912     .0066456  8.6412E-5


  S (A 14 COMPONENT ARRAY)
  9.7385  6.5841  6.4438  5.5233  3.802   3.1743  2.3796  2.1075  1.8207
  1.5919  1.1872  .83463  .51123  .74939
 THE MAXIMUM SINGULAR VALUE:
MAX =  9.7385
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .51123

 ...AND THEIR RATIO:
RTIO =  19.049


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     195.7138
FULL STATE ERRORS:     171.4433       24.2706

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     187.8448
CHANGE IN STATE ERRORS:     145.1516       42.6933

TERM:          X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:      90.6012
CHANGE IN STATE ERRORS:      82.8124        7.7888

TERM:          U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:       8.0872
CHANGE IN STATE ERRORS:       0.0621        8.0251

TERM:          U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:      39.0776
CHANGE IN STATE ERRORS:      39.0775        0.0001

TERM:       X1.X1.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:       0.0306
```

Figure 5.1.1b  Full Second Degree Model and Reduction Test (First Pass)

```
CHANGE IN STATE ERRORS:      -0.0430       0.0736

TERM:       X1.X2,  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:      -2.0255
CHANGE IN STATE ERRORS:      -2.0145      -0.0111

TERM:       X2.X2,  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:      -0.6560
CHANGE IN STATE ERRORS:      -0.6514      -0.0046

TERM:       X1.U1,  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:    -122.4182
CHANGE IN STATE ERRORS:    -122.3798      -0.0384

TERM:       X1.U2,  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:      -0.0639
CHANGE IN STATE ERRORS:       0.0211      -0.0849

TERM:       X2.U1,  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:       0.9927
CHANGE IN STATE ERRORS:       0.9675       0.0253

TERM:       X2.U2,  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:       0.2700
CHANGE IN STATE ERRORS:       0.1913       0.0787

TERM:       U1.U1,  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -15.4397
CHANGE IN STATE ERRORS:       0.2414     -15.6811

TERM:       U1.U2,  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:      -0.0333
CHANGE IN STATE ERRORS:      -0.0192      -0.0141

TERM:       U2.U2,  COLUMN #:  14.
CHANGE IN SYSTEM ERROR:      -0.1777
CHANGE IN STATE ERRORS:      -0.1797       0.0019

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 6

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
WHEN ASKED, ENTER A AS AN ARRAY.
A = ARRAY(6;1,2,3,4,8,12)

MANUAL MODE
 SPACE USED  63 K NOW,  72 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:Y
```

Figure 5.1.1c  Full Second Degree Model and Reduction Test (First Pass)

$$x_0 = (0.005, -0.005).$$

The control inputs are again given by

$$u_1(t) = 0.025\sin(2\pi t\cdot\phi_1) + 0.025\sin(2\pi t\cdot 2\phi_1),$$

$$u_2(t) = 0.025\sin(2\pi t\cdot\phi_2) + 0.025\sin(2\pi t\cdot 2\phi_2).$$

Now as starting frequencies for the optimization, we use the frequency set determined by the first pass optimimation, $\phi = (0.7418, 0.9022)$. The optimum frequencies are computed to be $\phi = (0.6499, 0.9014)$. Note that there is not much change in the optimum frequencies which is what we desire. The model is re-calculated and the reduction test again performed. See Figure 5.1.2 for the "second pass" output. Inspection of the reduction data indicates that the remaining terms are all fairly significant. So the model given in Figure 5.1.2 is the final reduced model. It contains 6 terms of a possible 14; a 57% reduction. Furthermore, by taking all the partial derivatives, the exact second degree approximation would be

$$L_{10} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad L_{01} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$L_{20} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_{11} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_{02} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Obviously, the reduction method singled out the appropriate columns to be kept. In addition the error test for the individual states indicates the correct insignificant entries in each column. The reduction method seems to

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0050     -0.0050

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2           2

ENTER THE   4 INITIAL INPUT AMPLITUDES:
 0.24999999999999996E-01  0.24999999999999996E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.7418      0.9022

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2    200   2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.6872968D+00  0.9476237D+00
THE CONDITION NUMBER:  0.126D+05
OBJECTIVE FUNCTION VALUE: 0.1975D+00

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.6782293D+00  0.9468474D+00
THE CONDITION NUMBER:  0.118D+05
OBJECTIVE FUNCTION VALUE: 0.1918D+00

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.6495584D+00  0.8921068D+00
THE CONDITION NUMBER:  0.107D+05
OBJECTIVE FUNCTION VALUE: 0.1745D+00
```

Figure 5.1.2a  Reduced Second Degree Model and Reduction Test (Second Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.6498634D+00  0.9013817D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

 50 CALLS OF FCN.....
THE PARAMETERS:
 0.6498652D+00  0.9013841D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

 60 CALLS OF FCN.....
THE PARAMETERS:
 0.6498653D+00  0.9013842D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS   6 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:       63

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.650     FREQ(2)=  0.901     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



TSO SPEAKEASY III PI+  7:33 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 6 COMPONENT ARRAY)
  1   2   3   4   8   12


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -2.003   -3.0012
  .9999    1.0001

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50146+.86391 -.50146-.86391


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  2.3743E-4  .9991
 -1.0005     -1.5637E-5
```

Figure 5.1.2b  Reduced Second Degree Model and Reduction Test (Second Pass)

```
PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  0  0  0
  0  0  0


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0049   0        0        0
  .016344  0        0        0


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.016019  0        0
 -.99954   0        0


  S (A 6 COMPONENT ARRAY)
  7.3342  2.012   2.2395  4.5216  5.5657  6.1157
 THE MAXIMUM SINGULAR VALUE:
MAX =  7.3342
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  2.012

 ...AND THEIR RATIO:
RTIO =  3.6452


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     200.9931
FULL STATE ERRORS:     176.7734       24.2198

TERM:          X1.  COLUMN #:  1.
CHANGE IN SYSTEM ERROR:    236.6909
CHANGE IN STATE ERRORS:    194.4798      42.2111

TERM:          X2.  COLUMN #:  2.
CHANGE IN SYSTEM ERROR:     67.8517
CHANGE IN STATE ERRORS:     60.9716       6.8801

TERM:          U1.  COLUMN #:  3.
CHANGE IN SYSTEM ERROR:     11.0693
CHANGE IN STATE ERRORS:      0.0157      11.0535

TERM:          U2.  COLUMN #:  4.
CHANGE IN SYSTEM ERROR:    -16.1350
CHANGE IN STATE ERRORS:    -16.1348      -0.0002

TERM:       X1.U1.  COLUMN #:  8.
CHANGE IN SYSTEM ERROR:   -120.4635
CHANGE IN STATE ERRORS:   -120.5174       0.0539

TERM:       U1.U1.  COLUMN #: 12.
CHANGE IN SYSTEM ERROR:    -15.9942
CHANGE IN STATE ERRORS:      0.2470     -16.2412

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N

MANUAL MODE
 SPACE USED  43 K NOW,  47 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.2c  Reduced Second Degree Model and Reduction Test (Second Pass)

have done about as well as could be expected. The true proof of a model's validity though lies in its simulation.

For the purpose of model verification, simulation consists of the excitation of the true system and the model at points about the point of identification. The control input excitation in this case is a single cosinusoid. That is

$$u_i(t) = \alpha_i \cos(2\pi t \cdot \phi_i) \quad i = 1,2,\ldots,m.$$

Furthermore, an initial condition for each state is required. The true system and each model are run for a specified time and sampled at evenly spaced points throughout the interval. If two models are simulated simultaneously, a mean square error (with respect to the true solution) analysis is performed. The error criterion, then, is just the ratio of the mean square errors of the two models for each state. For the following simulations, the error ratio is

$$R_i = \frac{\mathrm{MSE}(\mathrm{MODEL1}_i)}{\mathrm{MSE}(\mathrm{LINEAR}_i)}, \quad i = 1,2,\ldots,n$$

where in this case $\mathrm{MODEL1}_i$ refers to the i-th state of the six term reduced model. Obviously then, we desire the error ratios to be less than one. But, the following tabular data will show for some cases the error ratio between two models becomes very large. In these simulations it is necessary to check the normalized error data and closely examine the graphical output. Most of the time the large numbers are due to the fact that one model's trajectory lies on top of the true solution trajectory. Despite the fact that the other model may also be very close to the true solution, the error ratio becomes large.

Recall that two goals were set for performance evaluation of reduced models. The first is to outperform models of lesser degree, and the second is

to approximate full model behavior. Tables 5.1.1 through Table 5.1.8 addre the first criterion. In each of these tables, the heading MODEL1 refers to the reduced second degree model. The first three tables (Table 5.1.1 through Table 5.1.3) originally appear in [2] and test the model performance very close to the identification point. Note that throughout the trio of tables, according to the error ratios, the reduced model seems to be outperformed when the input amplitudes are zero. This behavior of the error criterion was explained above and a representative plot set is given in Figure 5.1.3.[1] Representative plot sets of the simulation data tables follow each table. Note that even with the loss of terms, the second degree model exhibits exceptional behavior over the linear approximation. In Table 5.1.4, the model's region of validity is tested farther away from the origin as the input amplitudes are increased substantially over those in previous tables. Table 5.1.5 gives a set of simulations whose initial state conditions are (-0.01, -0.01), and whose input amplitudes and frequencies are randomly chosen in the ranges (-0.2, 0.2) and (2, 6) respectively. The following three tables (Tables 5.1.6 through Table 5.1.8) test the low frequency and d.c. behavior of the models. Table 5.1.6 has various small frequencies and steps. Note that it was previously determined in [2] that the true system is unstable in the first state when excited with steps of magnitude more than 0.1. Table 5.1.7 is another random table of smaller steps whose initial state conditions are in the range (-0.01, 0.01) and whose amplitudes are in the range (-0.025, 0.025). Table 5.1.8 has some larger steps with the initial state conditions randomly chosen in the range (-0.05, 0.05) and input amplitudes also randomly chosen in the range (-0.075, 0.075).

---

[1]In all plots, if a curve is indistinguishable, it over-lays another curve.

********************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
********************************************

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.748E+06 | 0.606E+06 |
| 2 | 0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.408E-05 | 0.549E-04 |
| 3 | 0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.269E-05 | 0.452E-04 |
| 4 | 0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.[illegible]1E-05 | 0.362E-04 |
| 5 | 0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.638E-05 | 0.729E-04 |
| 6 | 0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.257E-03 | 0.162E-02 |
| 7 | 0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.258E-03 | 0.237E-02 |
| 8 | 0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.485E-03 | 0.318E-02 |
| 9 | 0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.702E-03 | 0.367E-02 |
| 10 | 0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.588E+07 | 0.759E+07 |
| 11 | 0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.277E-05 | 0.335E-04 |
| 12 | 0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.218E-05 | 0.374E-04 |
| 13 | 0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.121E-05 | 0.194E-04 |
| 14 | 0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.545E-05 | 0.568E-04 |
| 15 | 0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.320E-03 | 0.188E-02 |
| 16 | 0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.285E-03 | 0.251E-02 |
| 17 | 0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.581E-03 | 0.362E-02 |
| 18 | 0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.719E-03 | 0.371E-02 |
| 19 | -0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.748E+06 | 0.606E+06 |
| 20 | -0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.206E-05 | 0.256E-04 |
| 21 | -0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.190E-05 | 0.330E-04 |
| 22 | -0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.870E-06 | 0.148E-04 |
| 23 | -0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.582E-05 | 0.568E-04 |
| 24 | -0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.353E-03 | 0.202E-02 |
| 25 | -0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.295E-03 | 0.259E-02 |
| 26 | -0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.632E-03 | 0.387E-02 |
| 27 | -0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.759E-03 | 0.384E-02 |
| 28 | -0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.588E+07 | 0.759E+07 |
| 29 | -0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.381E-05 | 0.432E-04 |
| 30 | -0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.240E-05 | 0.409E-04 |
| 31 | -0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.187E-05 | 0.259E-04 |
| 32 | -0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.673E-05 | 0.729E-04 |
| 33 | -0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.286E-03 | 0.174E-02 |
| 34 | -0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.267E-03 | 0.244E-02 |
| 35 | -0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.53[illegible]-03 | 0.340E-02 |
| 36 | -0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.741E-03 | 0.379E-02 |

Table 5.1.1 Simulation Table for Linear Model versus Second Degree Reduced Model

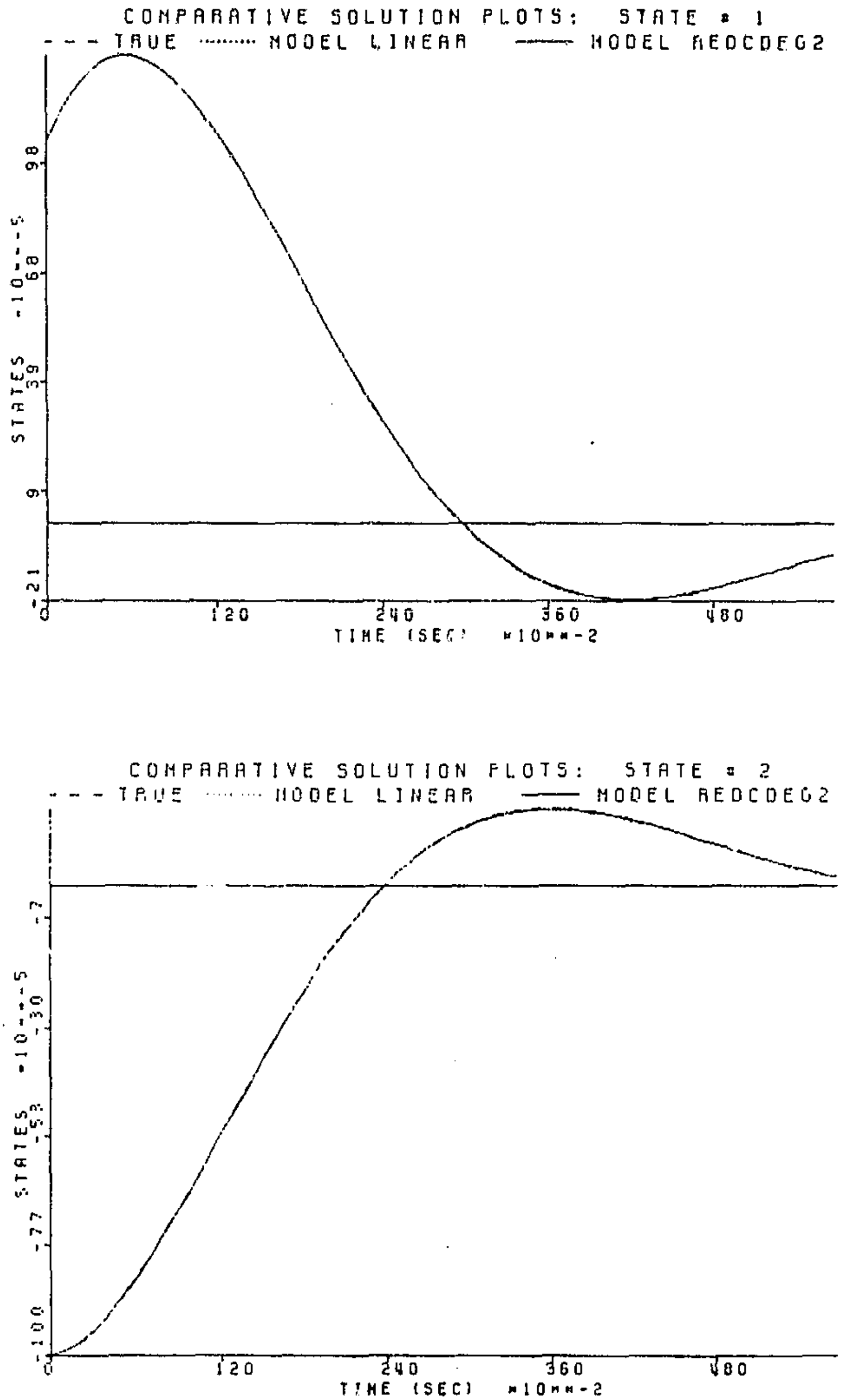
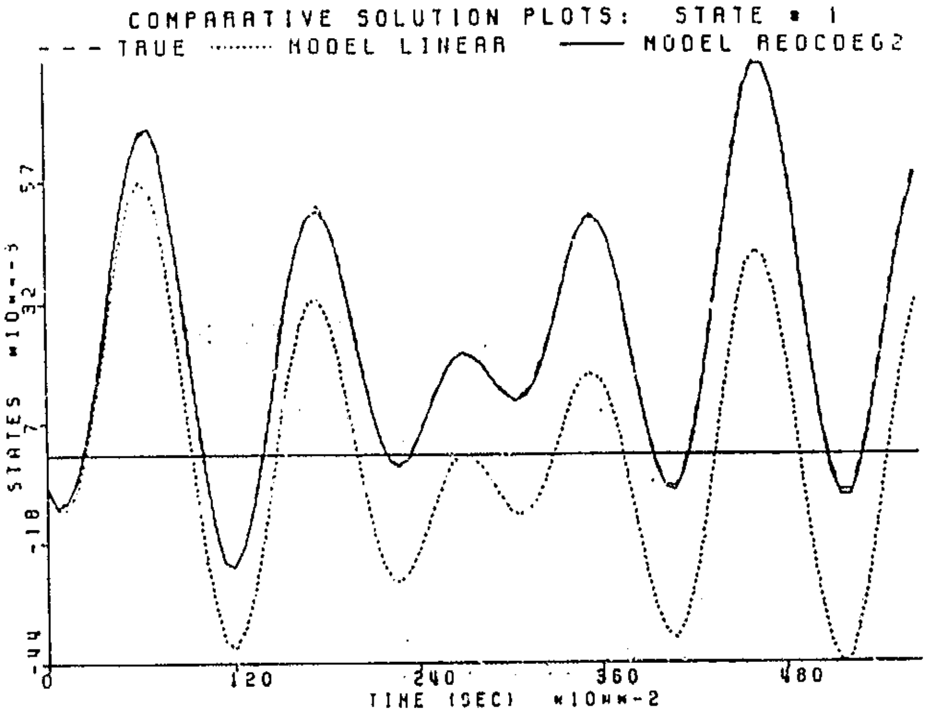



Figure 5.1.3 Simulation Number 10 of Table 5.1.1



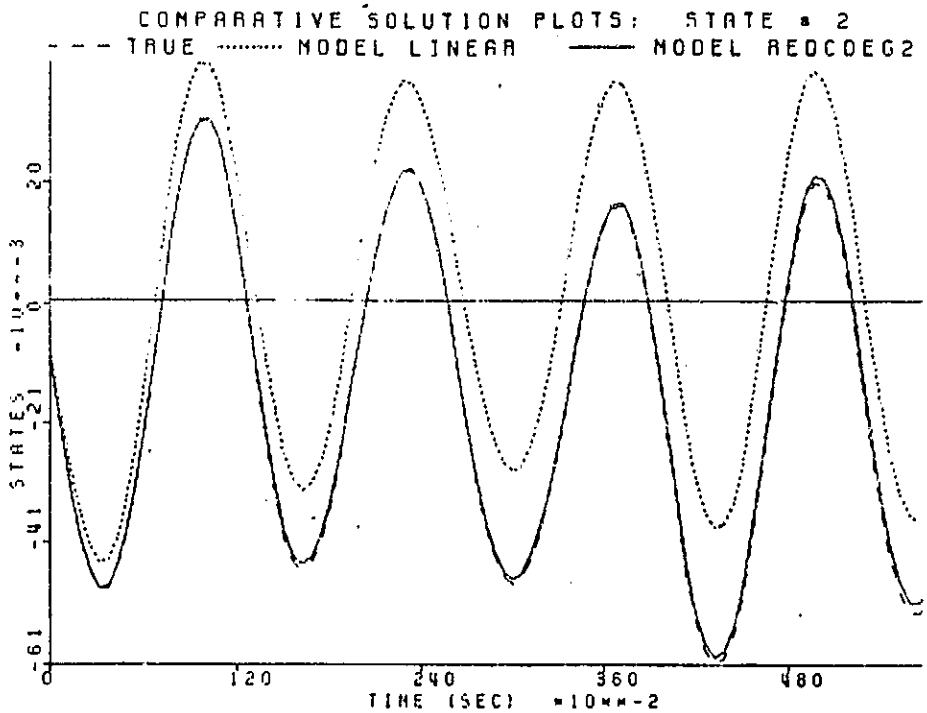




Figure 5.1.4 Simulation Number 25 of Table 5.1.1

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:    6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
*************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.005 | 0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 805. | 891. |
| 2 | 0.005 | 0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.162E-04 | 0.170E-03 |
| 3 | 0.005 | 0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.510E-05 | 0.811E-04 |
| 4 | 0.005 | 0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.118E-04 | 0.159E-03 |
| 5 | 0.005 | 0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.826E-05 | 0.116E-03 |
| 6 | 0.005 | 0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.133E-03 | 0.103E-02 |
| 7 | 0.005 | 0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.205E-03 | 0.203E-02 |
| 8 | 0.005 | 0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.269E-03 | 0.208E-02 |
| 9 | 0.005 | 0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.619E-03 | 0.344E-02 |
| 10 | 0.005 | -0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.704E+04 | 0.818E+04 |
| 11 | 0.005 | -0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.159E-05 | 0.217E-04 |
| 12 | 0.005 | -0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.222E-05 | 0.396E-04 |
| 13 | 0.005 | -0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.856E-06 | 0.172E-04 |
| 14 | 0.005 | -0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.342E-05 | 0.327E-04 |
| 15 | 0.005 | -0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.404E-03 | 0.219E-02 |
| 16 | 0.005 | -0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.331E-03 | 0.270E-02 |
| 17 | 0.005 | -0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.698E-03 | 0.413E-02 |
| 18 | 0.005 | -0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.688E-03 | 0.359E-02 |
| 19 | -0.005 | -0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 805. | 891. |
| 20 | -0.005 | -0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.191E-05 | 0.225E-04 |
| 21 | -0.005 | -0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.102E-05 | 0.181E-04 |
| 22 | -0.005 | -0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.338E-05 | 0.531E-04 |
| 23 | -0.005 | -0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.549E-05 | 0.332E-04 |
| 24 | -0.005 | -0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.615E-03 | 0.305E-02 |
| 25 | -0.005 | -0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.394E-03 | 0.314E-02 |
| 26 | -0.005 | -0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.101E-02 | 0.553E-02 |
| 27 | -0.005 | -0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.905E-03 | 0.429E-02 |
| 28 | -0.005 | 0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.704E+04 | 0.818E+04 |
| 29 | -0.005 | 0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.685E-05 | 0.703E-04 |
| 30 | -0.005 | 0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.333E-05 | 0.574E-04 |
| 31 | -0.005 | 0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.416E-05 | 0.499E-04 |
| 32 | -0.005 | 0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.996E-05 | 0.115E-03 |
| 33 | -0.005 | 0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.233E-03 | 0.152E-02 |
| 34 | -0.005 | 0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.244E-03 | 0.237E-02 |
| 35 | -0.005 | 0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.444E-03 | 0.299E-02 |
| 36 | -0.005 | 0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.794E-03 | 0.401E-02 |

Table 5.1.2 Simulation Table for Linear Model versus Second Degree Reduced Model

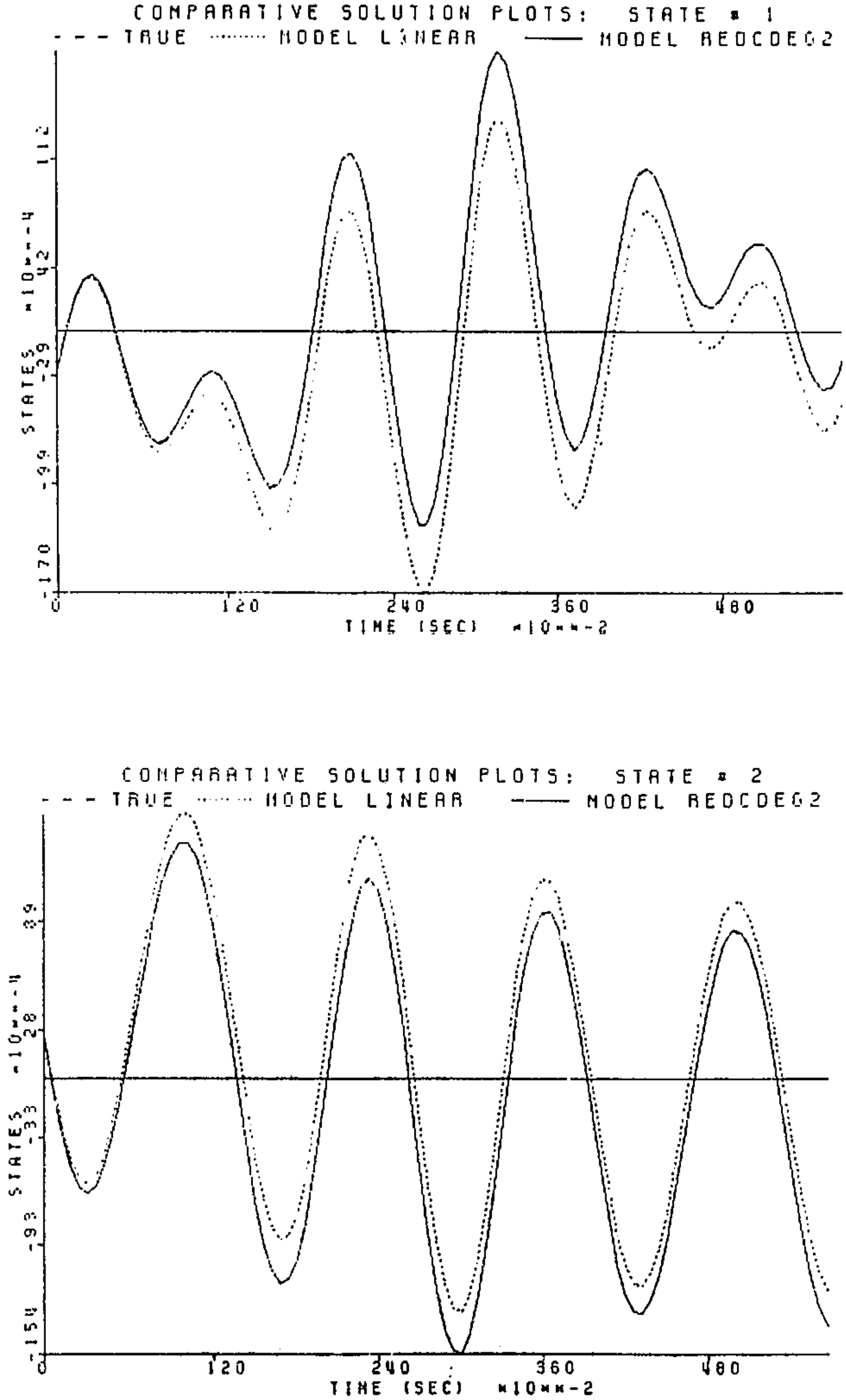
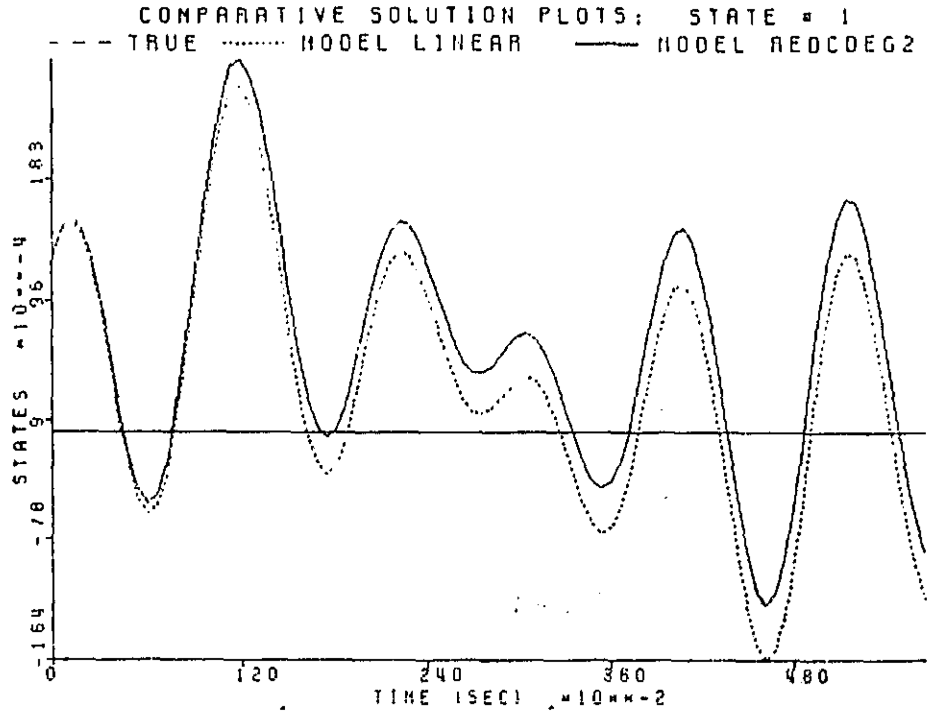



Figure 5.1.5 Simulation Number 29 of Table 5.1.2

```
************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.010 | 0.010 | 0.000 | 0.000 | 0.75 | 1.00 | 39.0 | 42.4 |
| 2 | 0.010 | 0.010 | 0.050 | 0.050 | 0.75 | 1.00 | 0.439E-04 | 0.446E-03 |
| 3 | 0.010 | 0.010 | 0.050 | -0.050 | 0.75 | 1.00 | 0.110E-04 | 0.166E-03 |
| 4 | 0.010 | 0.010 | -0.050 | -0.050 | 0.75 | 1.00 | 0.364E-04 | 0.507E-03 |
| 5 | 0.010 | 0.010 | -0.050 | 0.050 | 0.75 | 1.00 | 0.133E-04 | 0.208E-03 |
| 6 | 0.010 | 0.010 | 0.150 | 0.150 | 0.75 | 1.00 | 0.104E-03 | 0.692E-03 |
| 7 | 0.010 | 0.010 | 0.150 | -0.150 | 0.75 | 1.00 | 0.180E-03 | 0.178E-02 |
| 8 | 0.010 | 0.010 | -0.150 | -0.150 | 0.75 | 1.00 | 0.150E-03 | 0.126E-02 |
| 9 | 0.010 | 0.010 | -0.150 | 0.150 | 0.75 | 1.00 | 0.570E-03 | 0.334E-02 |
| 10 | 0.010 | -0.010 | 0.000 | 0.000 | 0.75 | 1.00 | 447. | 514. |
| 11 | 0.010 | -0.010 | 0.050 | 0.050 | 0.75 | 1.00 | 0.203E-05 | 0.229E-04 |
| 12 | 0.010 | -0.010 | 0.050 | -0.050 | 0.75 | 1.00 | 0.341E-05 | 0.635E-04 |
| 13 | 0.010 | -0.010 | -0.050 | -0.050 | 0.75 | 1.00 | 0.239E-05 | 0.365E-04 |
| 14 | 0.010 | -0.010 | -0.050 | 0.050 | 0.75 | 1.00 | 0.180E-05 | 0.173E-04 |
| 15 | 0.010 | -0.010 | 0.150 | 0.150 | 0.75 | 1.00 | 0.539E-03 | 0.267E-02 |
| 16 | 0.010 | -0.010 | 0.150 | -0.150 | 0.75 | 1.00 | 0.412E-03 | 0.306E-02 |
| 17 | 0.010 | -0.010 | -0.150 | -0.150 | 0.75 | 1.00 | 0.873E-03 | 0.486E-02 |
| 18 | 0.010 | -0.010 | -0.150 | 0.150 | 0.75 | 1.00 | 0.669E-03 | 0.354E-02 |
| 19 | -0.010 | -0.010 | 0.000 | 0.000 | 0.75 | 1.00 | 39.0 | 42.4 |
| 20 | -0.010 | -0.010 | 0.050 | 0.050 | 0.75 | 1.00 | 0.143E-04 | 0.148E-03 |
| 21 | -0.010 | -0.010 | 0.050 | -0.050 | 0.75 | 1.00 | 0.184E-05 | 0.265E-04 |
| 22 | -0.010 | -0.010 | -0.050 | -0.050 | 0.75 | 1.00 | 0.211E-04 | 0.289E-03 |
| 23 | -0.010 | -0.010 | -0.050 | 0.050 | 0.75 | 1.00 | 0.760E-05 | 0.283E-04 |
| 24 | -0.010 | -0.010 | 0.150 | 0.150 | 0.75 | 1.00 | 0.108E-02 | 0.476E-02 |
| 25 | -0.010 | -0.010 | 0.150 | -0.150 | 0.75 | 1.00 | 0.566E-03 | 0.403E-02 |
| 26 | -0.010 | -0.010 | -0.150 | -0.150 | 0.75 | 1.00 | 0.165E-02 | 0.816E-02 |
| 27 | -0.010 | -0.010 | -0.150 | 0.150 | 0.75 | 1.00 | 0.115E-02 | 0.507E-02 |
| 28 | -0.010 | 0.010 | 0.000 | 0.000 | 0.75 | 1.00 | 447. | 514. |
| 29 | -0.010 | 0.010 | 0.050 | 0.050 | 0.75 | 1.00 | 0.125E-04 | 0.120E-03 |
| 30 | -0.010 | 0.010 | 0.050 | -0.050 | 0.75 | 1.00 | 0.549E-05 | 0.972E-04 |
| 31 | -0.010 | 0.010 | -0.050 | -0.050 | 0.75 | 1.00 | 0.914E-05 | 0.105E-03 |
| 32 | -0.010 | 0.010 | -0.050 | 0.050 | 0.75 | 1.00 | 0.156E-04 | 0.192E-03 |
| 33 | -0.010 | 0.010 | 0.150 | 0.150 | 0.75 | 1.00 | 0.196E-03 | 0.132E-02 |
| 34 | -0.010 | 0.010 | 0.150 | -0.150 | 0.75 | 1.00 | 0.237E-03 | 0.237E-02 |
| 35 | -0.010 | 0.010 | -0.150 | -0.150 | 0.75 | 1.00 | 0.365E-03 | 0.257E-02 |
| 36 | -0.010 | 0.010 | -0.150 | 0.150 | 0.75 | 1.00 | 0.883E-03 | 0.437E-02 |

Table 5.1.3 Simulation Table for Linear Model versus Second Degree Reduced Model



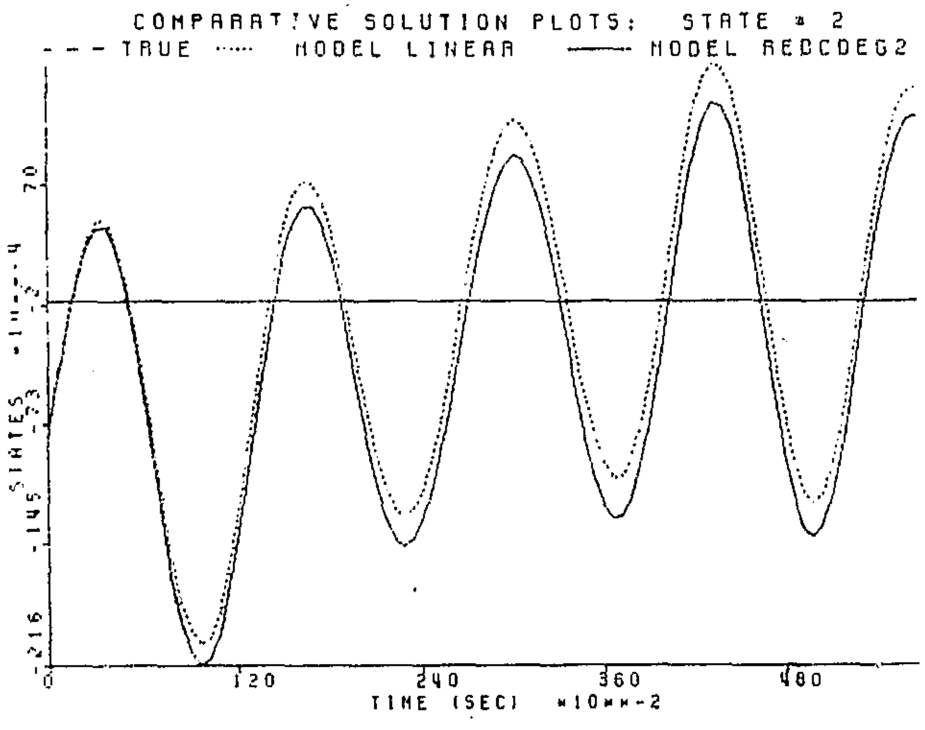
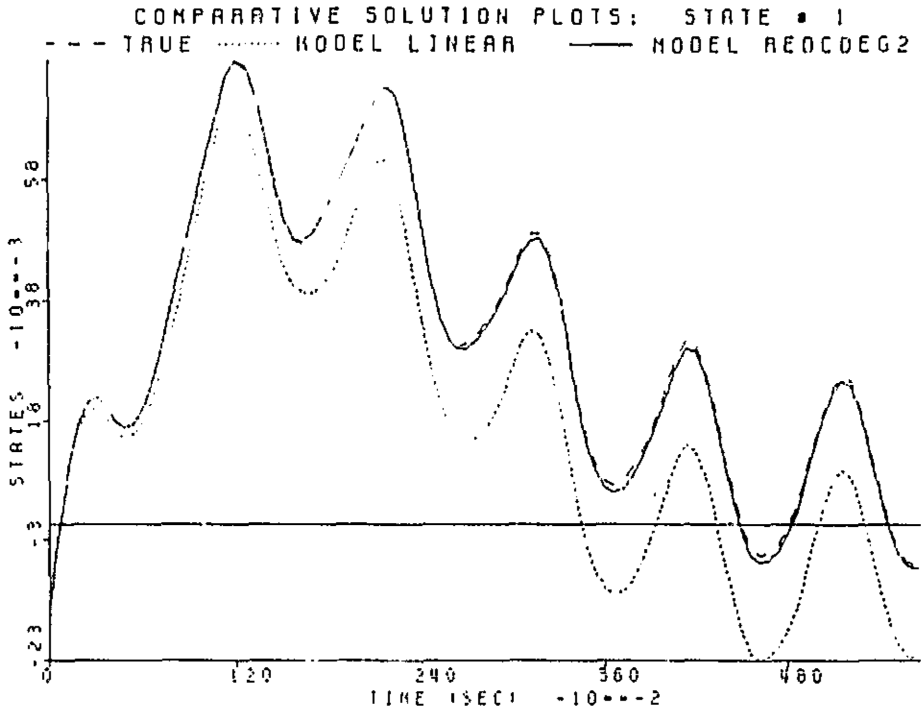




Figure 5.1.6 Simulation Number 14 of Table 5.1.3

```
**************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:    6
      DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.566E-03 | 0.231E-02 |
| 2 | 0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.497E-03 | 0.240E-02 |
| 3 | 0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.469E-03 | 0.195E-02 |
| 4 | 0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.414E-03 | 0.224E-02 |
| 5 | 0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.181E-03 | 0.866E-03 |
| 6 | 0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.204E-03 | 0.111E-02 |
| 7 | 0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.255E-03 | 0.115E-02 |
| 8 | 0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.275E-03 | 0.148E-02 |
| 9 | -0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.205E-03 | 0.862E-03 |
| 10 | -0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.151E-03 | 0.792E-03 |
| 11 | -0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.170E-03 | 0.806E-03 |
| 12 | -0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.136E-03 | 0.850E-03 |
| 13 | -0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.860E-03 | 0.359E-02 |
| 14 | -0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.906E-03 | 0.442E-02 |
| 15 | -0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.107E-02 | 0.424E-02 |
| 16 | -0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.104E-02 | 0.514E-02 |
| 17 | 0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.112E-02 | 0.513E-02 |
| 18 | 0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.778E-03 | 0.474E-02 |
| 19 | 0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.124E-02 | 0.610E-02 |
| 20 | 0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.912E-03 | 0.616E-02 |
| 21 | 0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.295E-02 | 0.101E-01 |
| 22 | 0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.297E-02 | 0.109E-01 |
| 23 | 0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.402E-02 | 0.126E-01 |
| 24 | 0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.380E-02 | 0.135E-01 |
| 25 | -0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.188E-02 | 0.759E-02 |
| 26 | -0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.155E-02 | 0.755E-02 |
| 27 | -0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.237E-02 | 0.931E-02 |
| 28 | -0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.195E-02 | 0.950E-02 |
| 29 | -0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.618E-02 | 0.170E-01 |
| 30 | -0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.620E-02 | 0.186E-01 |
| 31 | -0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.782E-02 | 0.204E-01 |
| 32 | -0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.734E-02 | 0.219E-01 |
| 33 | 0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.873E-02 | 0.347E-01 |
| 34 | 0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.776E-02 | 0.348E-01 |
| 35 | 0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.111E-01 | 0.411E-01 |
| 36 | 0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.940E-02 | 0.415E-01 |
| 37 | 0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.173E-01 | 0.500E-01 |
| 38 | 0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.169E-01 | 0.521E-01 |
| 39 | 0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.218E-01 | 0.589E-01 |
| 40 | 0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.200E-01 | 0.603E-01 |
| 41 | -0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.131E-01 | 0.437E-01 |
| 42 | -0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.121E-01 | 0.441E-01 |
| 43 | -0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.164E-01 | 0.510E-01 |
| 44 | -0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.143E-01 | 0.515E-01 |
| 45 | -0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.249E-01 | 0.627E-01 |
| 46 | -0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.244E-01 | 0.654E-01 |
| 47 | -0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.305E-01 | 0.729E-01 |
| 48 | -0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.281E-01 | 0.746E-01 |

Table 5.1.4 Simulation Table for Linear Model versus Second Degree Reduced Model



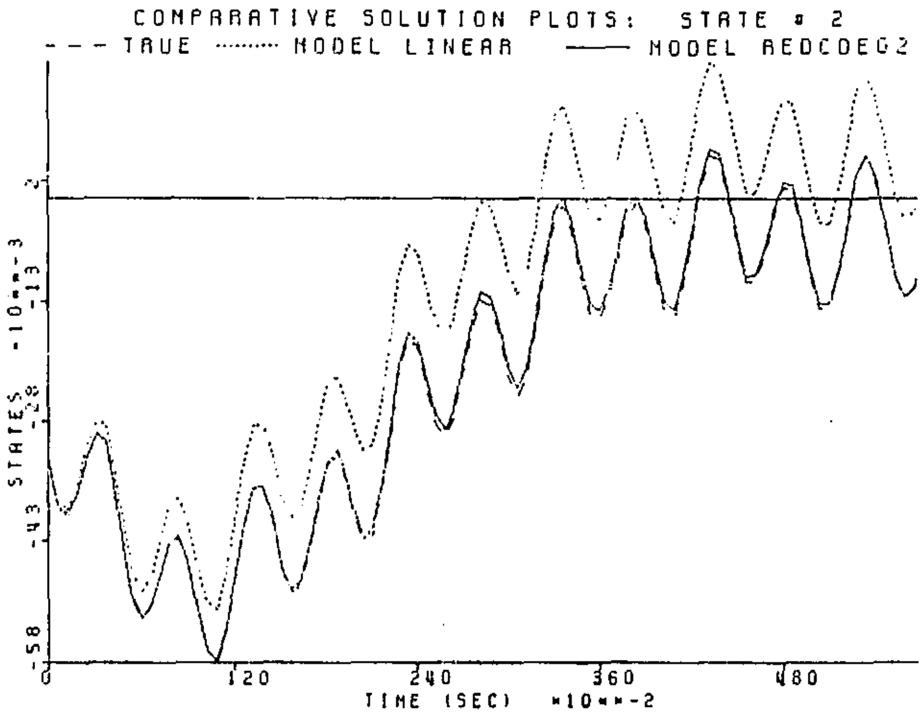
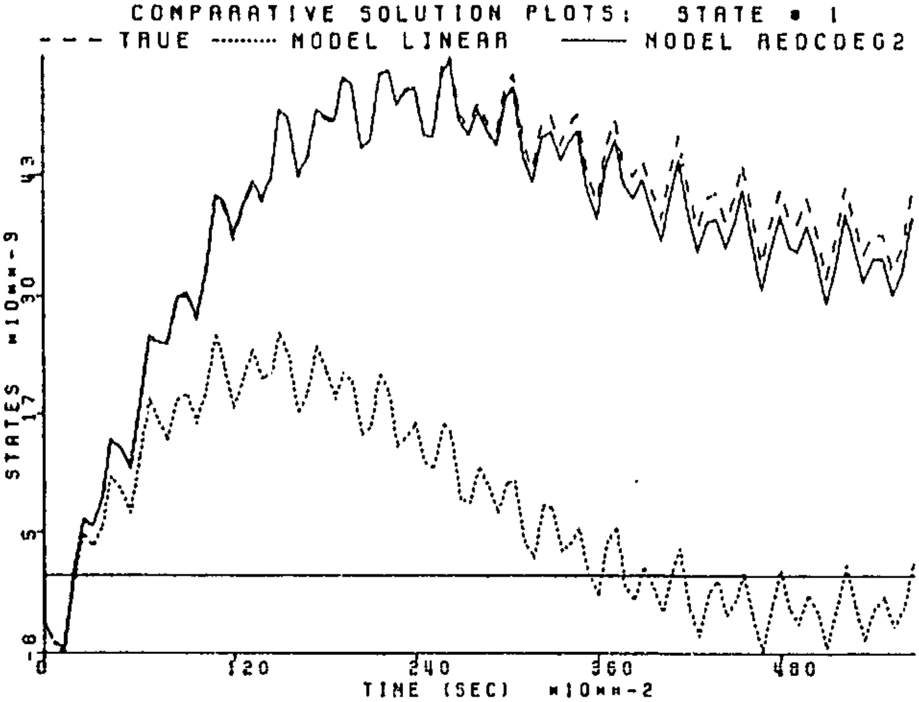




Figure 5.1.7 Simulation Number 13 of Table 5.1.4

```
*****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
*****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | -0.010 | -0.010 | 0.063 | 0.055 | 4.69 | 4.66 | 0.552E-04 | 0.315E-04 |
| 2 | -0.010 | -0.010 | 0.043 | -0.012 | 3.36 | 2.29 | 0.273E-04 | 0.417E-04 |
| 3 | -0.010 | -0.010 | 0.059 | 0.081 | 3.58 | 4.77 | 0.103E-03 | 0.215E-03 |
| 4 | -0.010 | -0.010 | 0.067 | 0.082 | 4.97 | 5.24 | 0.362E-03 | 0.648E-03 |
| 5 | -0.010 | -0.010 | -0.042 | 0.173 | 5.87 | 4.31 | 0.265E-05 | 0.408E-04 |
| 6 | -0.010 | -0.010 | 0.049 | -0.198 | 2.12 | 5.51 | 0.113E-04 | 0.970E-04 |
| 7 | -0.010 | -0.010 | -0.005 | -0.040 | 2.48 | 3.10 | 0.997E-04 | 0.216E-01 |
| 8 | -0.010 | -0.010 | 0.103 | 0.079 | 4.07 | 3.78 | 0.721E-03 | 0.137E-02 |
| 9 | -0.010 | -0.010 | -0.187 | 0.065 | 5.84 | 3.48 | 0.458E-02 | 0.974E-02 |
| 10 | -0.010 | -0.010 | -0.103 | -0.082 | 2.89 | 5.00 | 0.702E-03 | 0.125E-02 |
| 11 | -0.010 | -0.010 | -0.118 | -0.043 | 5.59 | 4.14 | 0.107E-02 | 0.193E-02 |
| 12 | -0.010 | -0.010 | -0.125 | -0.078 | 3.19 | 5.85 | 0.125E-02 | 0.233E-02 |
| 13 | -0.010 | -0.010 | 0.115 | 0.147 | 2.82 | 2.01 | 0.552E-03 | 0.170E-02 |
| 14 | -0.010 | -0.010 | 0.130 | 0.130 | 5.05 | 4.39 | 0.129E-02 | 0.263E-02 |
| 15 | -0.010 | -0.010 | 0.057 | 0.023 | 4.98 | 5.54 | 0.142E-03 | 0.193E-03 |
| 16 | -0.010 | -0.010 | -0.166 | 0.114 | 5.58 | 5.28 | 0.335E-02 | 0.644E-02 |
| 17 | -0.010 | [illegible] | -0.040 | 0.139 | 2.05 | 3.28 | 0.132E-05 | 0.265E-04 |
| 18 | -0.010 | -0.010 | -0.168 | 0.194 | 3.39 | 3.24 | 0.641E-03 | 0.296E-02 |
| 19 | -0.010 | -0.010 | -0.027 | 0.115 | 5.56 | 2.10 | 0.119E-05 | 0.733E-04 |
| 20 | -0.010 | -0.010 | 0.034 | 0.006 | 4.25 | 3.58 | 0.105E-04 | 0.322E-04 |
| 21 | -0.010 | -0.010 | 0.116 | -0.065 | 4.40 | 4.95 | 0.102E-02 | 0.182E-02 |
| 22 | -0.010 | -0.010 | -0.023 | 0.051 | 4.27 | 5.32 | 0.662E-04 | 0.900E-03 |
| 23 | -0.010 | -0.010 | -0.063 | -0.001 | 5.37 | 5.00 | 0.207E-03 | 0.274E-03 |
| 24 | -0.010 | -0.010 | -0.030 | -0.147 | 2.15 | 3.09 | 0.123E-05 | 0.115E-03 |
| 25 | -0.010 | -0.010 | 0.020 | -0.065 | 3.54 | 4.47 | 0.373E-04 | 0.177E-02 |
| 26 | -0.010 | -0.010 | -0.097 | 0.032 | 3.19 | 4.95 | 0.623E-03 | 0.101E-02 |
| 27 | -0.010 | -0.010 | -0.058 | 0.180 | 5.16 | 3.63 | 0.250E-04 | 0.201E-03 |
| 28 | -0.010 | -0.010 | 0.133 | 0.065 | 3.05 | 5.04 | 0.147E-02 | 0.281E-02 |
| 29 | -0.010 | -0.010 | -0.080 | -0.162 | 4.83 | 4.07 | 0.197E-03 | 0.590E-03 |
| 30 | -0.010 | -0.010 | -0.117 | -0.120 | 4.92 | 4.56 | 0.959E-03 | 0.193E-02 |
| 31 | -0.010 | -0.010 | -0.173 | 0.055 | 5.83 | 4.55 | 0.355E-02 | 0.738E-02 |
| 32 | -0.010 | -0.010 | -0.099 | -0.175 | 5.73 | 3.93 | 0.449E-03 | 0.111E-02 |
| 33 | -0.010 | -0.010 | -0.170 | -0.059 | 5.33 | 5.72 | 0.332E-02 | 0.687E-02 |
| 34 | -0.010 | -0.010 | -0.156 | -0.110 | 2.92 | 5.00 | 0.250E-02 | 0.511E-02 |
| 35 | -0.010 | -0.010 | 0.195 | -0.107 | 3.34 | 4.82 | 0.516E-02 | 0.112E-01 |
| 36 | -0.010 | -0.010 | 0.144 | 0.068 | 6.00 | 4.36 | 0.194E-02 | 0.380E-02 |
| 37 | -0.010 | -0.010 | -0.156 | 0.093 | 2.65 | 4.60 | 0.246E-02 | 0.503E-02 |
| 38 | -0.010 | -0.010 | 0.100 | -0.092 | 3.88 | 2.31 | 0.490E-03 | 0.114E-02 |
| 39 | -0.010 | -0.010 | -0.079 | 0.075 | 5.94 | 4.18 | 0.308E-03 | 0.536E-03 |
| 40 | -0.010 | -0.010 | -0.073 | -0.002 | 2.60 | 5.73 | 0.312E-03 | 0.441E-03 |
| 41 | -0.010 | -0.010 | 0.077 | 0.133 | 4.96 | 4.84 | 0.474E-03 | 0.848E-03 |
| 42 | -0.010 | -0.010 | 0.013 | 0.005 | 2.41 | 5.93 | 0.672E-02 | 0.153E-01 |
| 43 | -0.010 | -0.010 | 0.150 | -0.060 | 4.77 | 2.51 | 0.214E-02 | 0.434E-02 |
| 44 | -0.010 | -0.010 | 0.088 | -0.194 | 2.34 | 2.22 | 0.110E-03 | 0.754E-03 |
| 45 | -0.010 | -0.010 | 0.086 | 0.174 | 5.03 | 3.62 | 0.227E-03 | 0.706E-03 |
| 46 | -0.010 | -0.010 | 0.098 | -0.073 | 3.39 | 5.24 | 0.620E-03 | 0.106E-02 |
| 47 | -0.010 | -0.010 | -0.007 | -0.163 | 4.80 | 2.32 | 0.359E-05 | 0.427E-03 |
| 48 | -0.010 | -0.010 | 0.034 | -0.121 | 5.38 | 3.06 | 0.590E-06 | 0.268E-04 |
| 49 | -0.010 | -0.010 | 0.138 | -0.062 | 4.12 | 5.88 | 0.168E-02 | 0.323E-02 |
| 50 | -0.010 | -0.010 | -0.019 | -0.159 | 5.45 | 3.55 | 0.455E-05 | 0.996E-03 |

Table 5.1.5 Simulation Table for Linear Model versus Second Degree Reduced Model



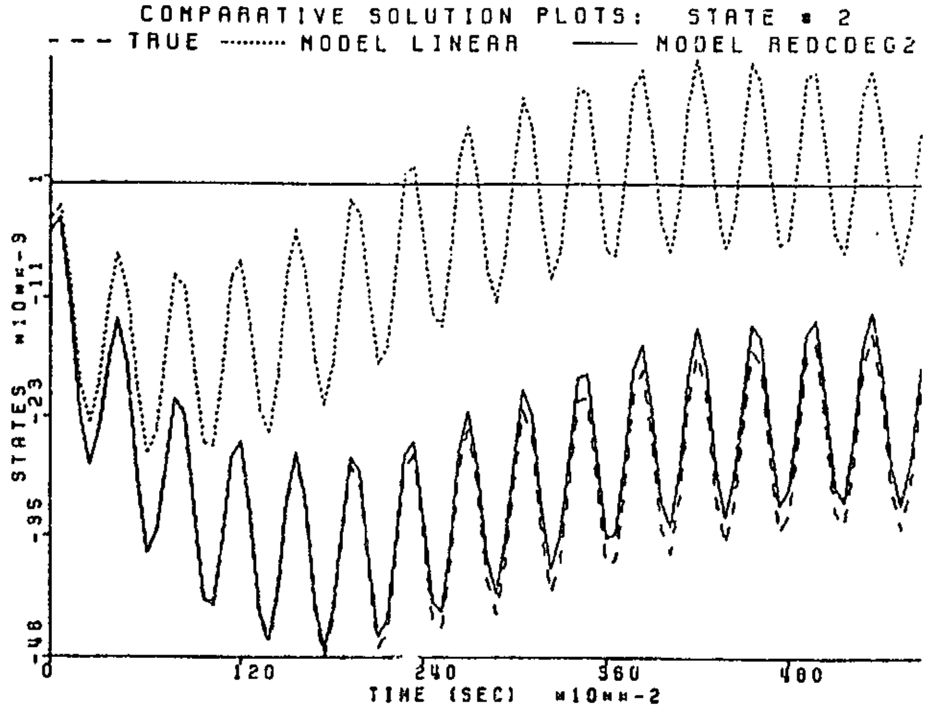
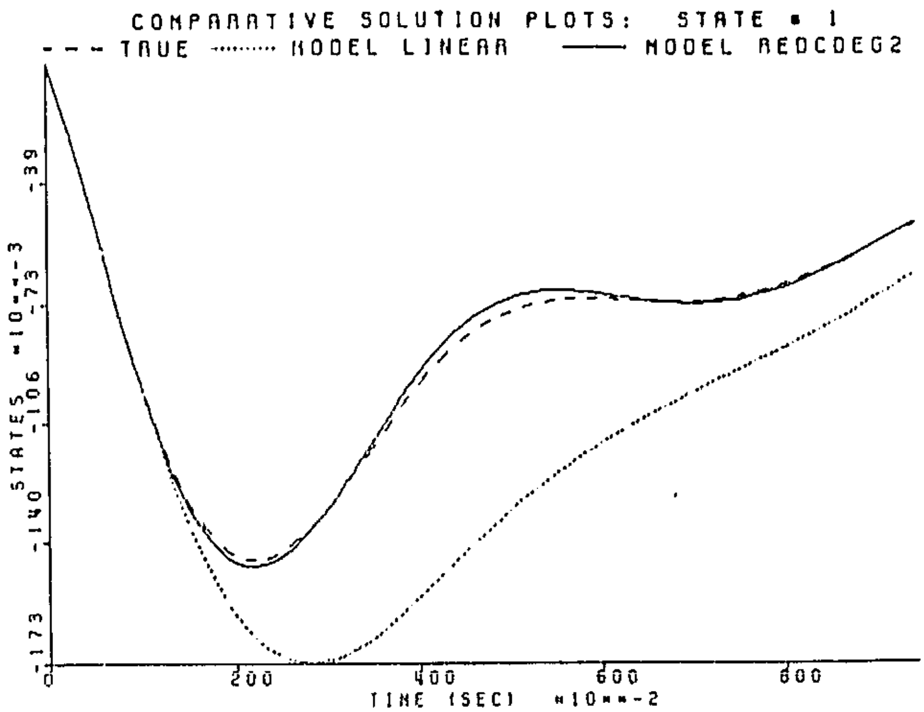




Figure 5.1.8 Simulation Number 37 of Table 5.1.5

```
***************************************
           PROBLEM SUMMARY
  CONFIGURATION:  TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:   6
      DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
***************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | -0.001 | 0.010 | -0.075 | 0.00 | 0.00 | 0.827E-04 | 0.769E-04 |
| 2 | 0.001 | -0.001 | 0.010 | -0.075 | 0.01 | 0.01 | 0.754E-04 | 0.695E-04 |
| 3 | 0.001 | -0.001 | 0.010 | -0.075 | 0.02 | 0.02 | 0.545E-04 | 0.527E-04 |
| 4 | 0.001 | -0.001 | 0.010 | -0.075 | 0.05 | 0.05 | 0.218E-04 | 0.180E-04 |
| 5 | 0.001 | -0.001 | -0.050 | 0.050 | 0.00 | 0.00 | 0.683E-02 | 0.594E-02 |
| 6 | 0.001 | -0.001 | -0.050 | 0.050 | 0.01 | 0.01 | 0.644E-02 | 0.548E-02 |
| 7 | 0.001 | -0.001 | -0.050 | 0.050 | 0.02 | 0.02 | 0.543E-02 | 0.486E-02 |
| 8 | 0.001 | -0.001 | -0.050 | 0.050 | 0.05 | 0.05 | 0.197E-02 | 0.165E-02 |
| 9 | 0.001 | -0.001 | -0.075 | 0.010 | 0.00 | 0.00 | 0.359E-01 | 0.459E-01 |
| 10 | 0.001 | -0.001 | -0.075 | 0.010 | 0.01 | 0.01 | 0.394E-01 | 0.487E-01 |
| 11 | 0.001 | -0.001 | -0.075 | 0.010 | 0.02 | 0.02 | 0.564E-01 | 0.663E-01 |
| 12 | 0.001 | -0.001 | -0.075 | 0.010 | 0.05 | 0.05 | 0.412E-01 | 0.216E-01 |
| 13 | 0.001 | -0.001 | -0.075 | -0.075 | 0.00 | 0.00 | 0.103E-03 | 0.115E-03 |
| 14 | 0.001 | -0.001 | -0.075 | -0.075 | 0.01 | 0.01 | 0.109E-03 | 0.111E-03 |
| 15 | 0.001 | -0.001 | -0.075 | -0.075 | 0.02 | 0.02 | 0.141E-03 | 0.124E-03 |
| 16 | 0.001 | -0.001 | -0.075 | -0.075 | 0.05 | 0.05 | 0.129E-0[illegible] | 0.181E-03 |
| 17 | 0.010 | -0.010 | 0.010 | -0.075 | 0.00 | 0.00 | 0.796E-0[illegible] | 0.747E-04 |
| 18 | 0.010 | -0.010 | 0.010 | -0.075 | 0.01 | 0.01 | 0.724E-04 | 0.673E-04 |
| 19 | 0.010 | -0.010 | 0.010 | -0.075 | 0.02 | 0.02 | 0.521E-04 | 0.507E-04 |
| 20 | 0.010 | -0.010 | 0.010 | -0.075 | 0.05 | 0.05 | 0.213E-04 | 0.177E-04 |
| 21 | 0.010 | -0.010 | -0.050 | 0.050 | 0.00 | 0.00 | 0.709E-02 | 0.608E-02 |
| 22 | 0.010 | -0.010 | -0.050 | 0.050 | 0.01 | 0.01 | 0.669E-02 | 0.563E-02 |
| 23 | 0.010 | -0.010 | -0.050 | 0.050 | 0.02 | 0.02 | 0.564E-02 | 0.500E-02 |
| 24 | 0.010 | -0.010 | -0.050 | 0.050 | 0.05 | 0.05 | 0.201E-02 | 0.167E-02 |
| 25 | 0.010 | -0.010 | -0.075 | 0.010 | 0.00 | 0.00 | 0.379E-01 | 0.474E-01 |
| 26 | 0.010 | -0.010 | -0.075 | 0.010 | 0.01 | 0.01 | 0.416E-01 | 0.506E-01 |
| 27 | 0.010 | -0.010 | -0.075 | 0.010 | 0.02 | 0.02 | 0.599E-01 | 0.699E-01 |
| 28 | 0.010 | -0.010 | -0.075 | 0.010 | 0.05 | 0.05 | 0.434E-01 | 0.220E-01 |
| 29 | 0.010 | -0.010 | -0.075 | -0.075 | 0.00 | 0.00 | 0.103E-03 | 0.114E-03 |
| 30 | 0.010 | -0.010 | -0.075 | -0.075 | 0.01 | 0.01 | 0.109E-03 | 0.109E-03 |
| 31 | 0.010 | -0.010 | -0.075 | -0.075 | 0.02 | 0.02 | 0.138E-03 | 0.121E-03 |
| 32 | 0.010 | -0.010 | -0.075 | -0.075 | 0.05 | 0.05 | 0.116E-03 | 0.173E-03 |

Table 5.1.6 Low Frequency Table for Linear Model versus Second Degree Reduced Model



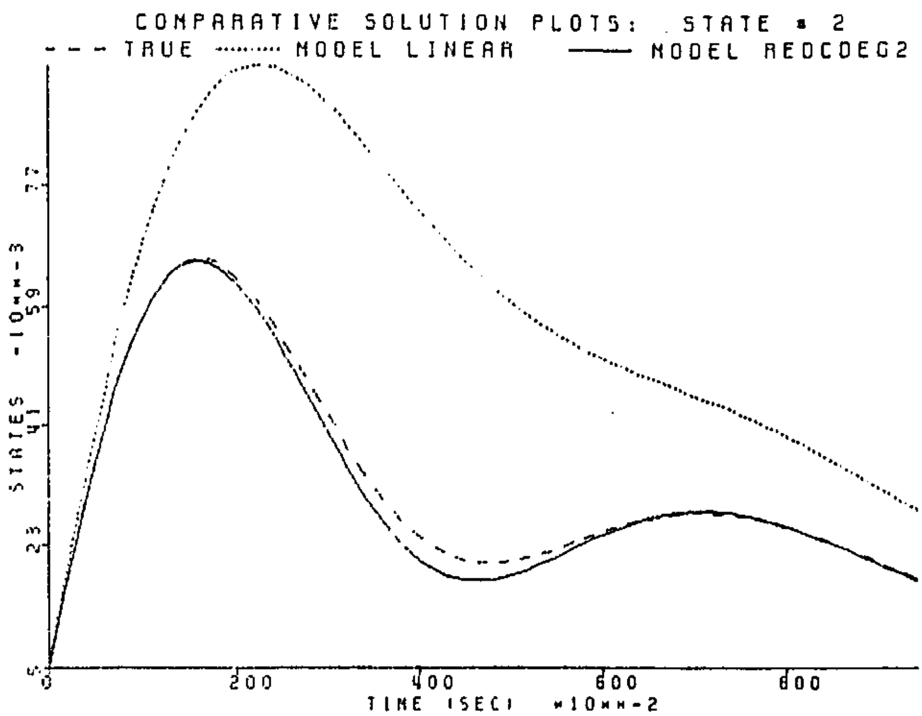




Figure 5.1.9 Simulation Number 15 of Table 5.1.6

```
**************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | -0.003 | 0.010 | 0.010 | -0.021 | 0.00 | 0.00 | 0.506E-05 | 0.488E-05 |
| 2 | 0.001 | -0.002 | -0.006 | -0.015 | 0.00 | 0.00 | 0.144E-05 | 0.112E-05 |
| 3 | 0.008 | 0.003 | 0.002 | 0.023 | 0.00 | 0.00 | 0.207E-06 | 0.123E-06 |
| 4 | 0.010 | -0.003 | 0.015 | -0.022 | 0.00 | 0.00 | 0.465E-04 | 0.464E-04 |
| 5 | -0.009 | 0.002 | 0.010 | 0.003 | 0.00 | 0.00 | 0.987E-04 | 0.687E-04 |
| 6 | 0.008 | 0.002 | -0.006 | -0.013 | 0.00 | 0.00 | 0.336E-05 | 0.249E-05 |
| 7 | 0.005 | 0.005 | 0.012 | -0.002 | 0.00 | 0.00 | 0.677E-05 | 0.252E-05 |
| 8 | 0.007 | 0.009 | 0.016 | 0.011 | 0.00 | 0.00 | 0.[illegible]11E-05 | 0.158E-05 |
| 9 | -0.005 | -0.004 | -0.002 | 0.002 | 0.00 | 0.00 | 0.242E-04 | 0.177E-04 |
| 10 | 0.009 | -0.000 | -0.008 | -0.018 | 0.00 | 0.00 | 0.224E-05 | 0.166E-05 |
| 11 | 0.006 | -0.005 | -0.012 | 0.011 | 0.00 | 0.00 | 0.240E-05 | 0.892E-06 |
| 12 | -0.007 | -0.003 | -0.004 | -0.014 | 0.00 | 0.00 | 0.294E-06 | 0.229E-06 |
| 13 | 0.000 | -0.001 | -0.008 | -0.007 | 0.00 | 0.00 | 0.153E-04 | 0.110E-04 |
| 14 | -0.010 | 0.008 | -0.015 | 0.003 | 0.00 | 0.00 | 0.431E-03 | 0.603E-04 |
| 15 | 0.009 | -0.005 | -0.002 | 0.004 | 0.00 | 0.00 | 0.856E-05 | 0.615E-05 |
| 16 | -0.004 | -0.001 | -0.001 | 0.006 | 0.00 | 0.00 | 0.151E-05 | 0.105E-05 |
| 17 | 0.001 | -0.007 | -0.005 | -0.008 | 0.00 | 0.00 | 0.318E-05 | 0.251E-05 |
| 18 | 0.001 | 0.003 | 0.009 | 0.008 | 0.00 | 0.00 | 0.127E-04 | 0.103E-04 |
| 19 | 0.005 | -0.004 | 0.009 | -0.024 | 0.00 | 0.00 | 0.296E-05 | 0.291E-05 |
| 20 | 0.009 | 0.009 | 0.002 | 0.016 | 0.00 | 0.00 | 0.167E-06 | 0.891E-07 |
| 21 | -0.008 | 0.003 | 0.006 | -0.003 | 0.00 | 0.00 | 0.262E-04 | 0.213E-04 |
| 22 | -0.009 | 0.008 | 0.021 | 0.023 | 0.00 | 0.00 | 0.155E-05 | 0.550E-06 |
| 23 | -0.003 | -0.007 | -0.002 | -0.021 | 0.00 | 0.00 | 0.110E-06 | 0.635E-07 |
| 24 | 0.008 | -0.002 | -0.011 | 0.016 | 0.00 | 0.00 | 0.983E-06 | 0.911E-06 |
| 25 | 0.003 | 0.000 | -0.023 | 0.001 | 0.00 | 0.00 | 0.877E-03 | 0.130E-02 |
| 26 | 0.000 | -0.005 | 0.014 | -0.024 | 0.00 | 0.00 | 0.413E-04 | 0.407E-04 |
| 27 | -0.004 | 0.008 | 0.014 | -0.024 | 0.00 | 0.00 | 0.518E-04 | 0.505E-04 |
| 28 | 0.006 | 0.008 | -0.011 | 0.024 | 0.00 | 0.00 | 0.416E-05 | 0.472E-05 |
| 29 | -0.006 | -0.003 | 0.016 | 0.014 | 0.00 | 0.00 | 0.363E-05 | 0.144E-05 |
| 30 | 0.001 | 0.009 | -0.019 | 0.010 | 0.00 | 0.00 | 0.104E-03 | 0.119E-03 |
| 31 | 0.003 | 0.008 | -0.025 | 0.022 | 0.00 | 0.00 | 0.343E-03 | 0.348E-03 |
| 32 | -0.010 | 0.004 | 0.009 | -0.024 | 0.00 | 0.00 | 0.446E-05 | 0.427E-05 |
| 33 | -0.002 | 0.002 | 0.020 | 0.021 | 0.00 | 0.00 | 0.161E-05 | 0.394E-06 |
| 34 | -0.009 | 0.006 | -0.021 | -0.009 | 0.00 | 0.00 | 0.922E-05 | 0.173E-05 |
| 35 | -0.008 | 0.004 | -0.011 | -0.005 | 0.00 | 0.00 | 0.435E-04 | 0.261E-04 |
| 36 | 0.005 | -0.008 | -0.017 | 0.003 | 0.00 | 0.00 | 0.280E-03 | 0.224E-03 |
| 37 | -0.006 | 0.008 | 0.007 | 0.006 | 0.00 | 0.00 | 0.167E-04 | 0.139E-04 |
| 38 | -0.008 | 0.008 | -0.017 | 0.018 | 0.00 | 0.00 | 0.313E-04 | 0.365E-04 |
| 39 | 0.000 | 0.004 | -0.016 | 0.018 | 0.00 | 0.00 | 0.225E-04 | 0.257E-04 |
| 40 | -0.005 | 0.006 | 0.019 | 0.012 | 0.00 | 0.00 | 0.663E-05 | 0.686E-05 |
| 41 | -0.008 | -0.004 | 0.020 | 0.017 | 0.00 | 0.00 | 0.532E-05 | 0.545E-05 |
| 42 | -0.007 | -0.009 | 0.014 | -0.012 | 0.00 | 0.00 | 0.320E-04 | 0.325E-04 |
| 43 | -0.006 | -0.001 | 0.022 | 0.006 | 0.00 | 0.00 | 0.332E-02 | 0.381E-02 |
| 44 | -0.008 | 0.003 | -0.017 | -0.012 | 0.00 | 0.00 | 0.980E-05 | 0.342E-05 |
| 45 | -0.005 | 0.005 | 0.007 | -0.007 | 0.00 | 0.00 | 0.417E-05 | 0.314E-05 |
| 46 | -0.004 | 0.002 | -0.005 | 0.003 | 0.00 | 0.00 | 0.666E-04 | 0.483E-04 |
| 47 | -0.004 | -0.005 | 0.020 | -0.022 | 0.00 | 0.00 | 0.281E-03 | 0.286E-03 |
| 48 | 0.003 | -0.009 | 0.015 | -0.004 | 0.00 | 0.00 | 0.694E-04 | 0.788E-04 |
| 49 | -0.007 | -0.010 | -0.001 | 0.011 | 0.00 | 0.00 | 0.251E-05 | 0.163E-05 |
| 50 | 0.007 | -0.003 | -0.006 | 0.001 | 0.00 | 0.00 | 0.430E-03 | 0.303E-03 |

Table 5.1.7 Step Response Table for Linear Model versus Second Degree Reduced Model

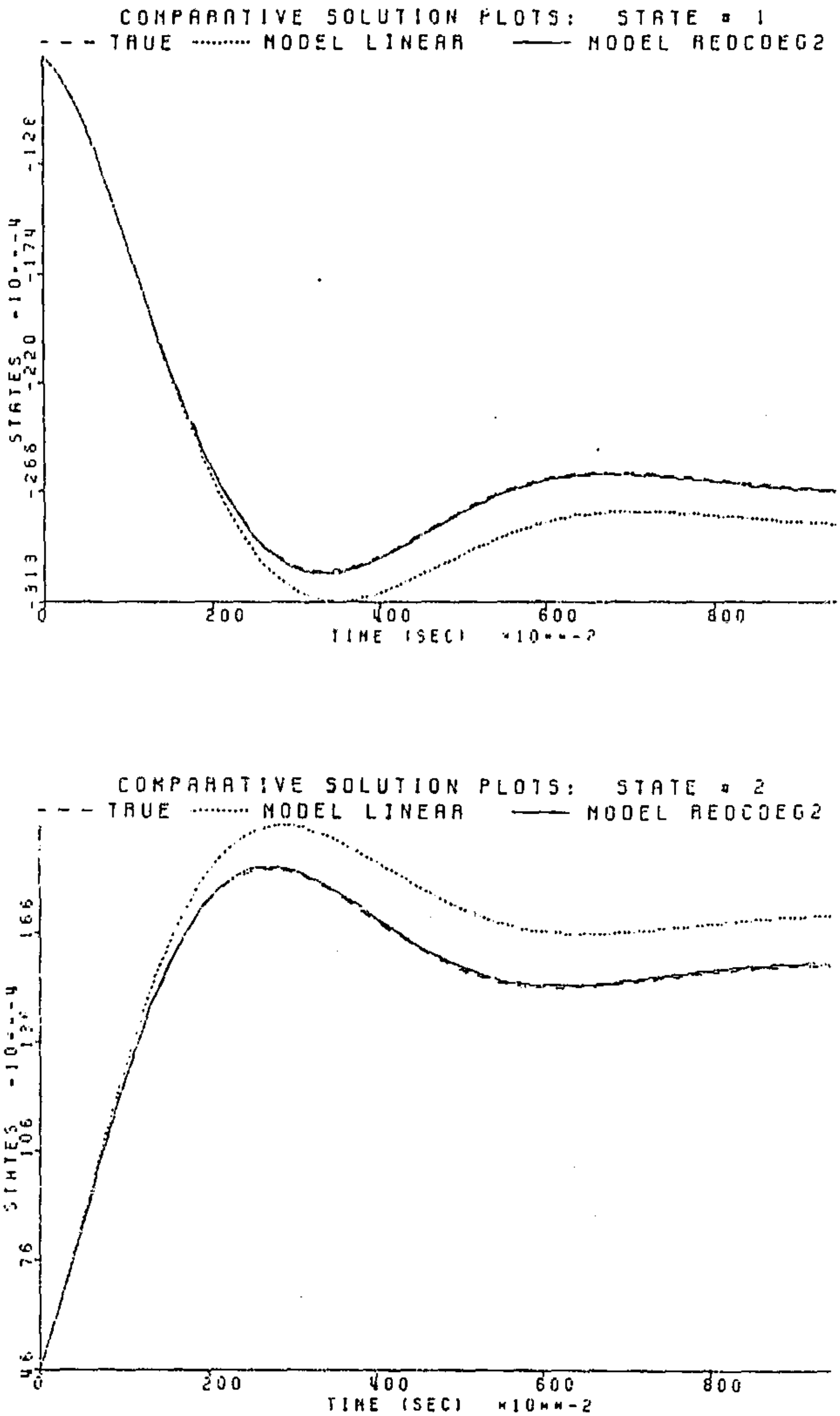
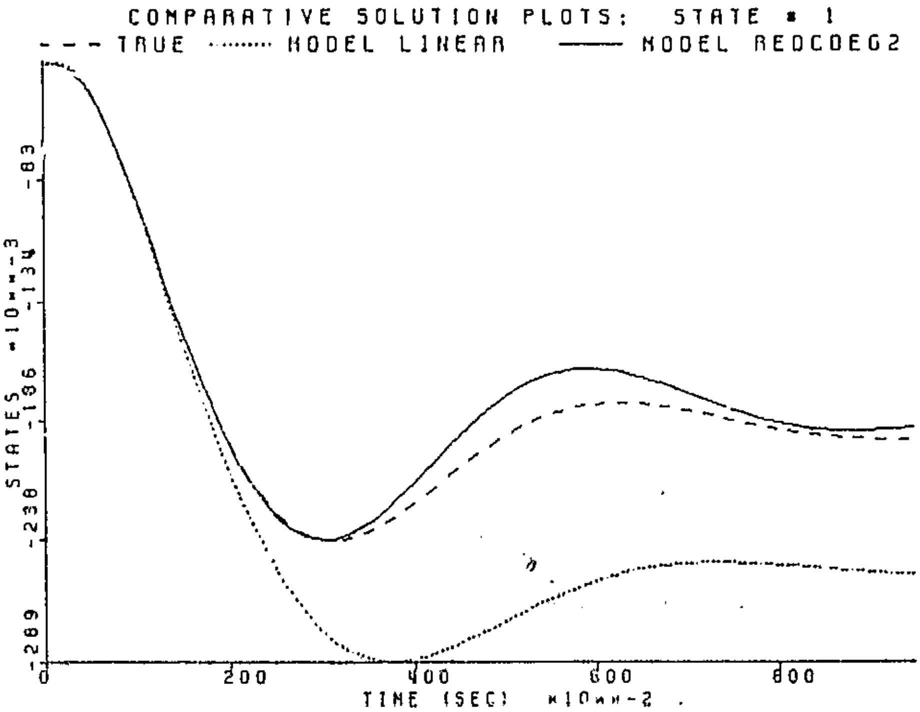



Figure 5.1.10 Simulation Number 35 of Table 5.1.7

***************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
***************************************

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | -0.022 | 0.031 | -0.062 | -0.034 | 0.00 | 0.00 | 0.451E-03 | 0.520E-03 |
| 2 | 0.010 | 0.011 | 0.005 | -0.050 | 0.00 | 0.00 | 0.617E-05 | 0.561E-05 |
| 3 | -0.044 | -0.004 | 0.055 | 0.063 | 0.00 | 0.00 | 0.339E-02 | 0.426E-02 |
| 4 | 0.015 | -0.006 | -0.007 | 0.042 | 0.00 | 0.00 | 0.350E-05 | 0.344[illegible] |
| 5 | 0.019 | 0.042 | 0.038 | -0.049 | 0.00 | 0.00 | 0.581E-02 | 0.636E-[illegible] |
| 6 | 0.023 | -0.029 | 0.068 | 0.027 | 0.00 | 0.00 | 0.260 | 0.346 |
| 7 | 0.048 | 0.035 | -0.027 | -0.052 | 0.00 | 0.00 | 0.391E-04 | 0.170E-04 |
| 8 | -0.006 | 0.016 | 0.024 | -0.027 | 0.00 | 0.00 | 0.724E-03 | 0.741E-03 |
| 9 | 0.026 | -0.043 | 0.049 | 0.044 | 0.00 | 0.00 | 0.622E-02 | 0.733E-02 |
| 10 | 0.018 | -0.016 | 0.054 | 0.071 | 0.00 | 0.00 | 0.648E-03 | 0.943E-03 |
| 11 | 0.029 | 0.042 | -0.018 | 0.066 | 0.00 | 0.00 | 0.240E-03 | 0.203E-03 |
| 12 | 0.011 | -0.001 | 0.049 | -0.064 | 0.00 | 0.00 | 0.223E-01 | 0.262E-01 |
| 13 | -0.027 | 0.022 | -0.003 | -0.045 | 0.00 | 0.00 | 0.400E-06 | 0.269E-06 |
| 14 | -0.028 | 0.030 | 0.033 | 0.029 | 0.00 | 0.00 | 0.190E-03 | 0.245E-03 |
| 15 | -0.025 | -0.002 | 0.068 | 0.056 | 0.00 | 0.00 | 0.143 | 0.240 |
| 16 | 0.006 | 0.010 | -0.015 | -0.014 | 0.00 | 0.00 | 0.862E-05 | 0.405E-05 |
| 17 | 0.011 | -0.001 | -0.036 | -0.057 | 0.00 | 0.00 | 0.357E-05 | 0.451E-06 |
| 18 | 0.011 | -0.005 | 0.070 | 0.004 | 0.00 | 0.00 | 0.140 | 0.180 |
| 19 | 0.021 | -0.044 | 0.010 | -0.010 | 0.00 | 0.00 | 0.309E-04 | 0.217E-04 |
| 20 | -0.026 | 0.030 | -0.013 | 0.053 | 0.00 | 0.00 | 0.277E-04 | 0.277E-04 |
| 21 | 0.035 | -0.021 | 0.008 | 0.059 | 0.00 | 0.00 | 0.356E-06 | 0.253E-06 |
| 22 | -0.024 | -0.040 | -0.018 | -0.046 | 0.00 | 0.00 | 0.106E-05 | 0.654E-06 |
| 23 | 0.047 | 0.036 | -0.022 | 0.064 | 0.00 | 0.00 | 0.482E-03 | 0.395E-03 |
| 24 | -0.027 | -0.018 | -0.018 | 0.047 | 0.00 | 0.00 | 0.109E-03 | 0.972E-04 |
| 25 | -0.038 | 0.015 | -0.014 | 0.012 | 0.00 | 0.00 | 0.526E-05 | 0.588E-05 |
| 26 | -0.004 | 0.050 | 0.070 | -0.068 | 0.00 | 0.00 | 0.130 | 0.184 |
| 27 | -0.008 | 0.026 | 0.050 | -0.020 | 0.00 | 0.00 | 0.261E-01 | 0.311E-01 |
| 28 | -0.036 | 0.020 | -0.064 | 0.041 | 0.00 | 0.00 | 0.406E-01 | 0.342E-01 |
| 29 | -0.000 | 0.047 | 0.021 | 0.070 | 0.00 | 0.00 | 0.279E-06 | 0.276E-06 |
| 30 | -0.027 | -0.004 | -0.033 | -0.005 | 0.00 | 0.00 | 0.393E-03 | 0.529E-03 |
| 31 | -0.025 | 0.035 | 0.047 | -0.012 | 0.00 | 0.00 | 0.225E-01 | 0.266E-01 |
| 32 | 0.029 | -0.024 | 0.062 | -0.020 | 0.00 | 0.00 | 0.709E-01 | 0.885E-01 |
| 33 | 0.041 | -0.038 | -0.014 | -0.053 | 0.00 | 0.00 | 0.179E-06 | 0.149E-06 |
| 34 | -0.042 | 0.018 | -0.016 | -0.069 | 0.00 | 0.00 | 0.611E-06 | 0.347E-06 |
| 35 | 0.038 | -0.006 | 0.022 | 0.063 | 0.00 | 0.00 | 0.124E-05 | 0.995E-06 |
| 36 | 0.049 | -0.015 | 0.075 | 0.005 | 0.00 | 0.00 | 0.173 | 0.227 |
| 37 | 0.012 | -0.036 | -0.033 | 0.053 | 0.00 | 0.00 | 0.112E-02 | 0.950E-03 |
| 38 | -0.017 | -0.006 | 0.023 | -0.033 | 0.00 | 0.00 | 0.591E-03 | 0.595E-03 |
| 39 | -0.039 | -0.043 | -0.049 | 0.037 | 0.00 | 0.00 | 0.942E-02 | 0.787E-02 |
| 40 | -0.006 | 0.007 | -0.055 | 0.067 | 0.00 | 0.00 | 0.785E-02 | 0.673E-02 |
| 41 | 0.032 | 0.016 | 0.040 | -0.011 | 0.00 | 0.00 | 0.106E-01 | 0.123E-01 |
| 42 | 0.030 | -0.011 | 0.030 | 0.016 | 0.00 | 0.00 | 0.936E-03 | 0.117E-02 |
| 43 | 0.009 | -0.018 | 0.043 | -0.026 | 0.00 | 0.00 | 0.120E-01 | 0.136E-01 |
| 44 | 0.015 | 0.001 | -0.015 | -0.009 | 0.00 | 0.00 | 0.135E-04 | 0.561E-05 |
| 45 | 0.034 | 0.042 | 0.063 | 0.003 | 0.00 | 0.00 | 0.873E-01 | 0.114 |
| 46 | 0.007 | -0.047 | 0.007 | 0.032 | 0.00 | 0.00 | 0.290E-05 | 0.953E-06 |
| 47 | 0.008 | -0.038 | 0.042 | 0.071 | 0.00 | 0.00 | 0.809E-04 | 0.762E-04 |
| 48 | -0.036 | 0.018 | -0.074 | 0.031 | 0.00 | 0.00 | 0.474 | 0.514 |
| 49 | 0.026 | 0.017 | 0.061 | 0.006 | 0.00 | 0.00 | 0.920E-01 | 0.119 |
| 50 | -0.029 | 0.038 | -0.023 | -0.055 | 0.00 | 0.00 | 0.254E-05 | 0.593E-06 |

Table 5.1.8 Step Response Table for Linear Model versus Second Degree Reduced Model



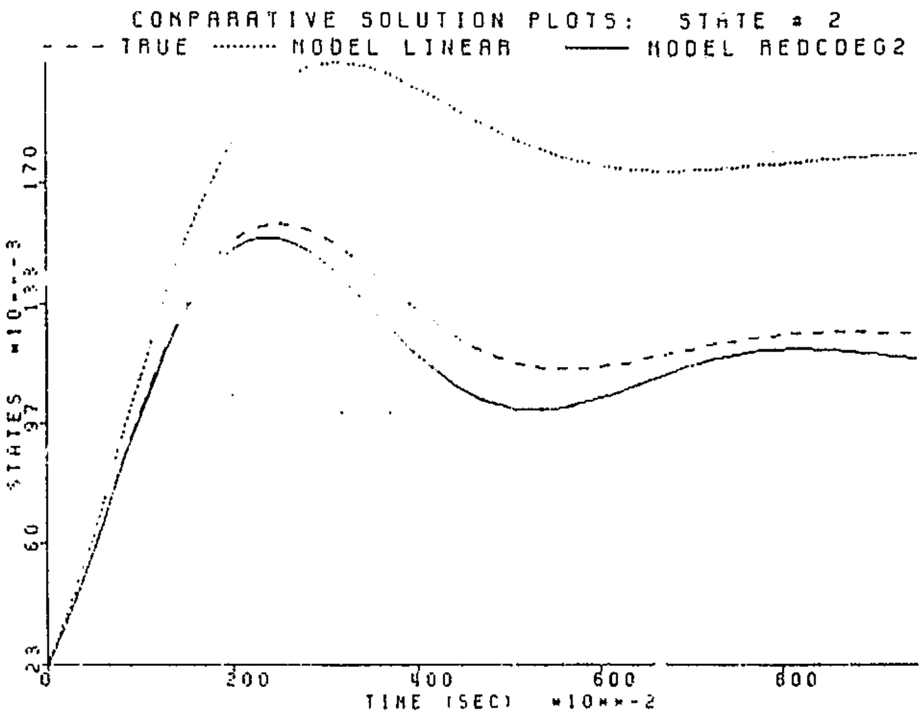




Figure 5.1.11 Simulation Number 48 of Table 5.1.8

From the tables and plots it is obvious that the second degree reduced model performs very well with respect to the linear model. Clearly, the loss of the eight terms in the model has not caused model performance to deteriorate significantly. In order to better check the redu d model's degradation of tracking ability, it is compared against the full second degree model. Tables 5.1.9 through 5.1.13 and the corresponding plots will show that good model behavior is preserved despite the loss of more than half the full model's terms. In each case now, we do not expect to outperform the full model, rather we just hope to approximate it. The error ratio, now

$$R_i = \frac{MSE(MODEL2_i)}{MSE(MODEL1_i)} \quad , \quad i = 1,2,\ldots,n,$$

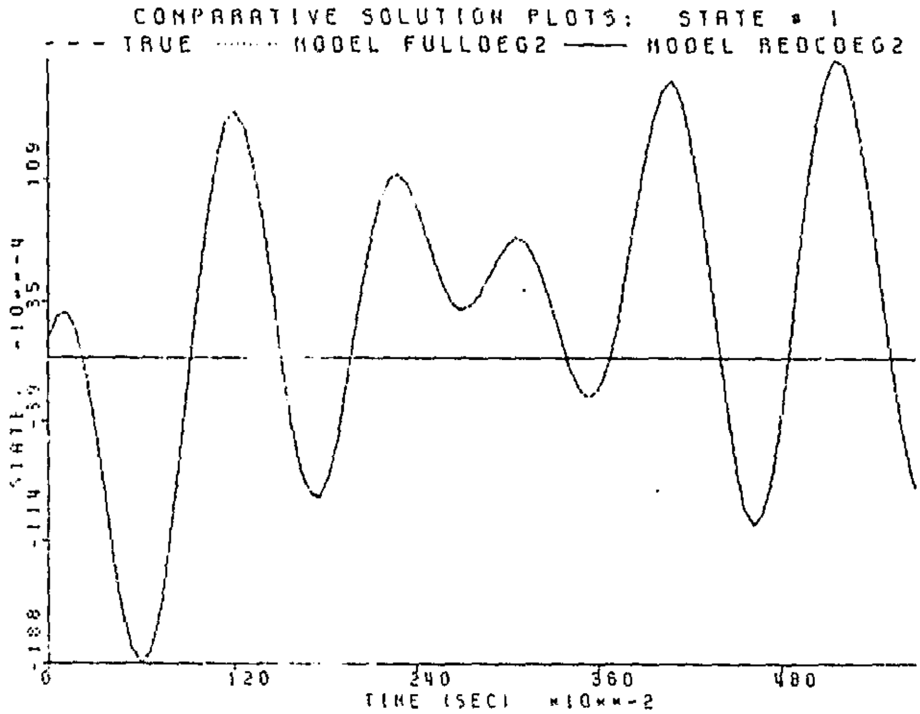
(where MODEL1 and MODEL2 refer to the full and reduced models resp ctively) is expected to be around unity. This may or may not be true, again depending on the relative closeness of a model's solution to the true solution, so we rely on the graphical data. Table 5.1.9 is a table of simulations close to the ori in. The error ratios meet their expectation and the plots in Figures 5.1.12 and 5.1.13 present typical curves. Table 5.1.10 moves the excitation parameters farther away from the origin. Figure 5.1.14 gives a representative set of plots, and Figure 5.1.15 is a blow up of Figure 5.1.14 to give a better look at the reduced model performance. Tables 5.1.11 and 5.1.2 are tables with constant initial state conditions and randomly chosen frequency and amplitude pairs. Table 5.1.11 has amplitudes in the range (-0.05, 0.05), and Table 5.1.12 has amplitudes in the range (-0.25, 0.25). Both have frequencies chosen from the range (2,6). Finally, Table 5.1.13 tests the low frequency and d.c. behavior of the full and reduced second degree models.

From all of the results presented so far it appears the model reduction technique is working quite well as both criteria for judging reduced models

```
***************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
***************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.713 | 0.722 |
| 2 | 0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.765 | 0.834 |
| 3 | 0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.278 | 0.475 |
| 4 | 0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.486 | 0.641 |
| 5 | 0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 2.40 | 2.95 |
| 6 | 0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.523 | 0.686 |
| 7 | 0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.661 | 0.762 |
| 8 | 0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.582 | 0.705 |
| 9 | 0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 1.15 | 0.944 |
| 10 | 0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 1.18 | 1.26 |
| 11 | 0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.554 | 0.738 |
| 12 | 0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.219 | 0.369 |
| 13 | 0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.246 | 0.426 |
| 14 | 0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 2.52 | 3.07 |
| 15 | 0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.561 | 0.706 |
| 16 | 0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.654 | 0.754 |
| 17 | 0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.608 | 0.720 |
| 18 | 0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 1.11 | 0.928 |
| 19 | -0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.899 | 0.910 |
| 20 | -0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.397 | 0.596 |
| 21 | -0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.219 | 0.369 |
| 22 | -0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.159 | 0.308 |
| 23 | -0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 3.30 | 3.71 |
| 24 | -0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.572 | 0.711 |
| 25 | -0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.667 | 0.758 |
| 26 | -0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.616 | 0.724 |
| 27 | -0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 1.10 | 0.926 |
| 28 | -0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 1.03 | 1.10 |
| 29 | -0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.605 | 0.715 |
| 30 | -0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.286 | 0.489 |
| 31 | -0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.345 | 0.493 |
| 32 | -0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 3.19 | 3.47 |
| 33 | -0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.536 | 0.692 |
| 34 | -0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.675 | 0.766 |
| 35 | -0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.591 | 0.710 |
| 36 | -0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 1.14 | 0.942 |

Table 5.1.9 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



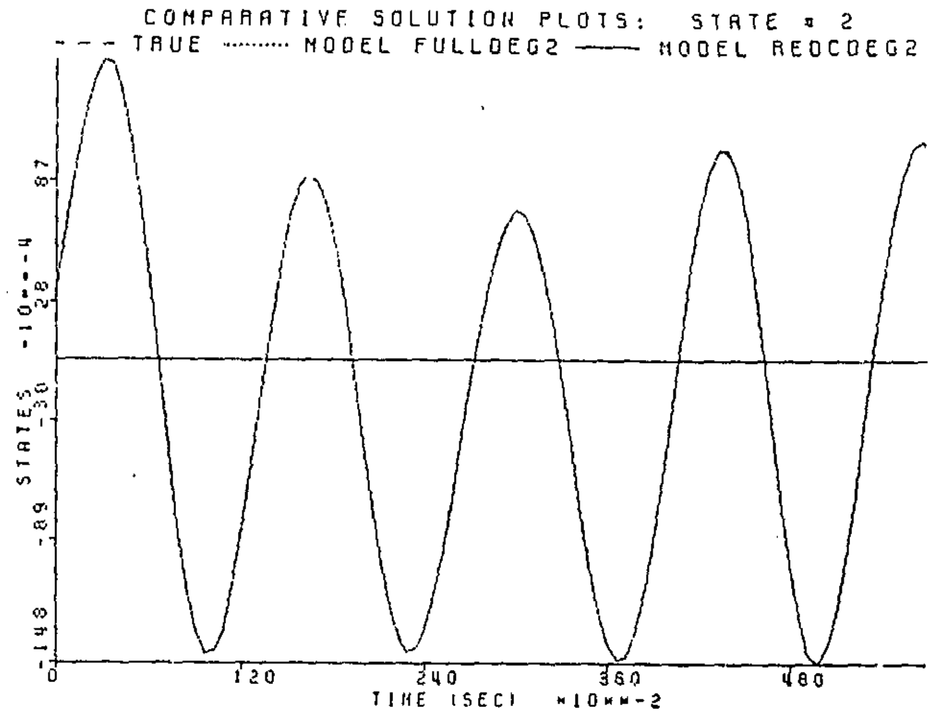




Figure 5.1.12 Simulation Number 32 of Table 5.1.9

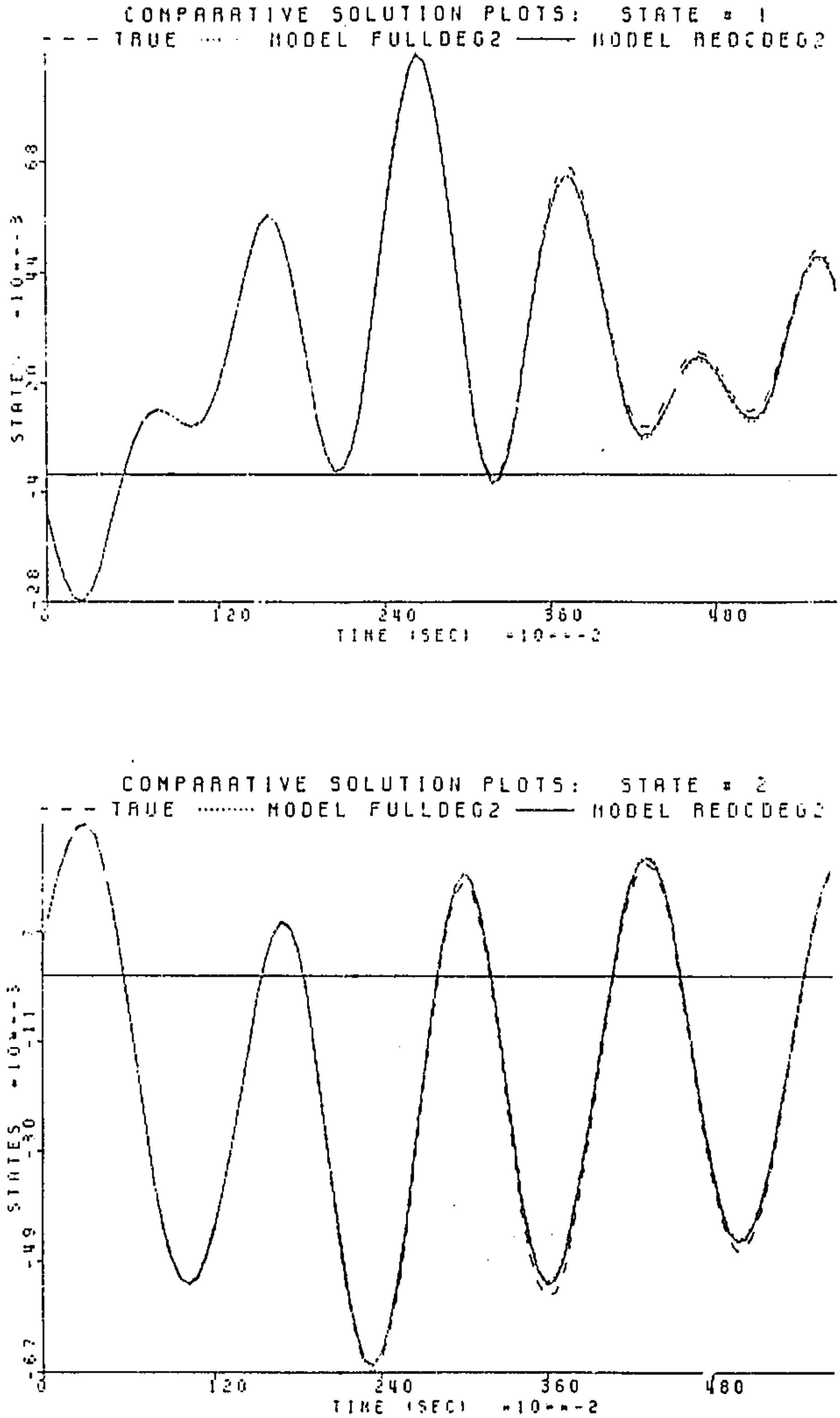



Figure 5.1.13 Simulation Number 26 of Table 5.1.9

```
**************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 1.21 | 1.21 |
| 2 | 0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 1.18 | 1.17 |
| 3 | 0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 1.22 | 1.22 |
| 4 | 0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 1.20 | 1.22 |
| 5 | 0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.876 | 0.936 |
| 6 | 0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.855 | 0.920 |
| 7 | 0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.924 | 0.979 |
| 8 | 0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.889 | 0.939 |
| 9 | -0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.978 | 0.987 |
| 10 | -0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.924 | 0.930 |
| 11 | -0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.992 | 1.00 |
| 12 | -0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 1.00 | 1.03 |
| 13 | -0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.717 | 0.743 |
| 14 | -0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.710 | 0.736 |
| 15 | -0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.729 | 0.756 |
| 16 | -0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.724 | 0.749 |
| 17 | 0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.913 | 0.932 |
| 18 | 0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.900 | 0.929 |
| 19 | 0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.924 | 0.940 |
| 20 | 0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.933 | 0.951 |
| 21 | 0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.929 | 0.949 |
| 22 | 0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.932 | 0.952 |
| 23 | 0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.936 | 0.953 |
| 24 | 0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.946 | 0.959 |
| 25 | -0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.907 | 0.931 |
| 26 | -0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.909 | 0.934 |
| 27 | -0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.916 | 0.936 |
| 28 | -0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.937 | 0.950 |
| 29 | -0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.880 | 0.907 |
| 30 | -0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.880 | 0.906 |
| 31 | -0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.885 | 0.910 |
| 32 | -0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.892 | 0.914 |
| 33 | 0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.892 | 0.915 |
| 34 | 0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.898 | 0.918 |
| 35 | 0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.905 | 0.923 |
| 36 | 0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.916 | 0.929 |
| 37 | 0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.907 | 0.925 |
| 38 | 0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.911 | 0.927 |
| 39 | 0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.915 | 0.930 |
| 40 | 0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.923 | 0.935 |
| 41 | -0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.892 | 0.915 |
| 42 | -0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.896 | 0.917 |
| 43 | -0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.902 | 0.921 |
| 44 | -0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.912 | 0.927 |
| 45 | -0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.896 | 0.917 |
| 46 | -0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.898 | 0.918 |
| 47 | -0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.901 | 0.921 |
| 48 | -0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.908 | 0.926 |

Table 5.1.10 Simulation Tables for Second Degree Full Model versus Second Degree Reduced Model

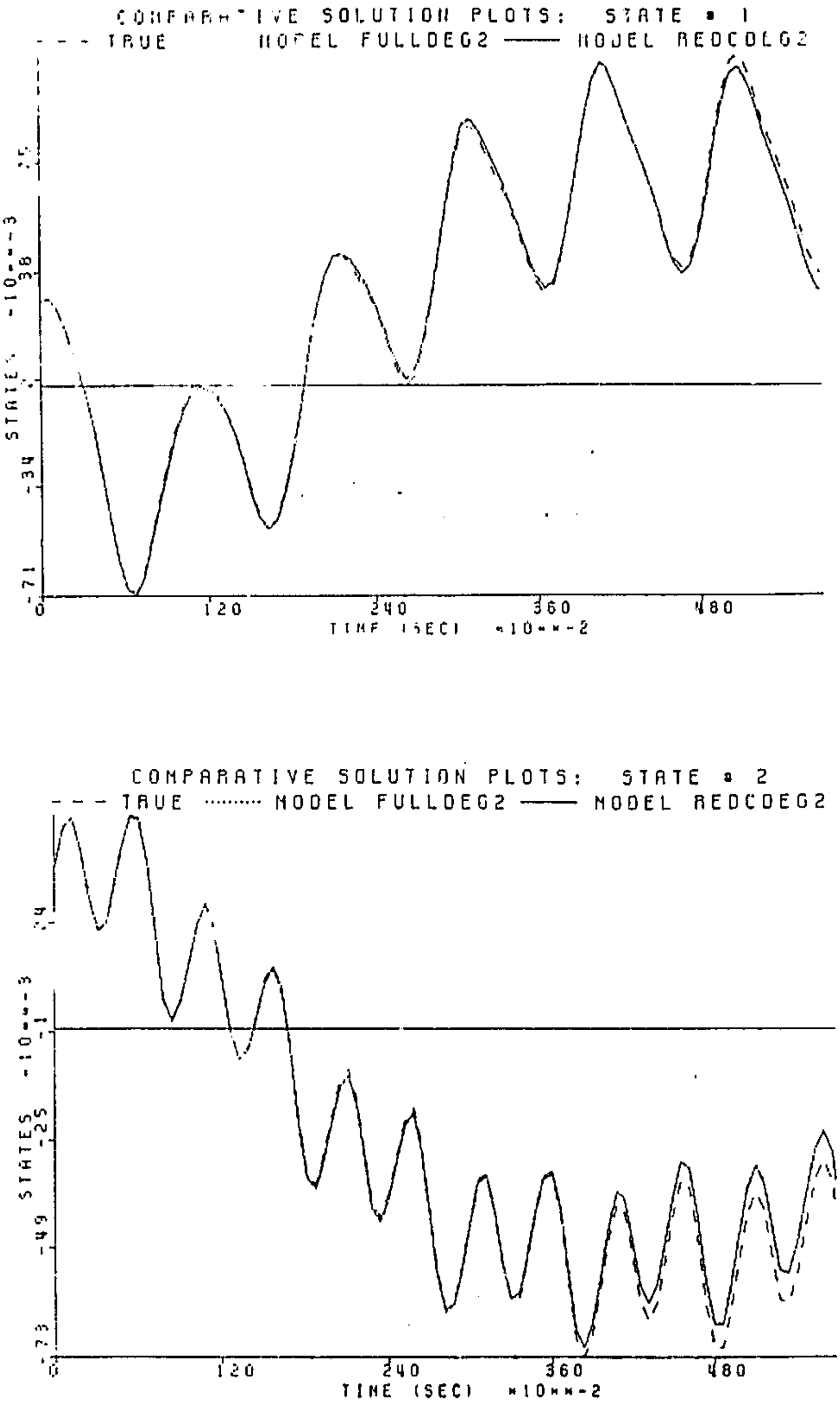
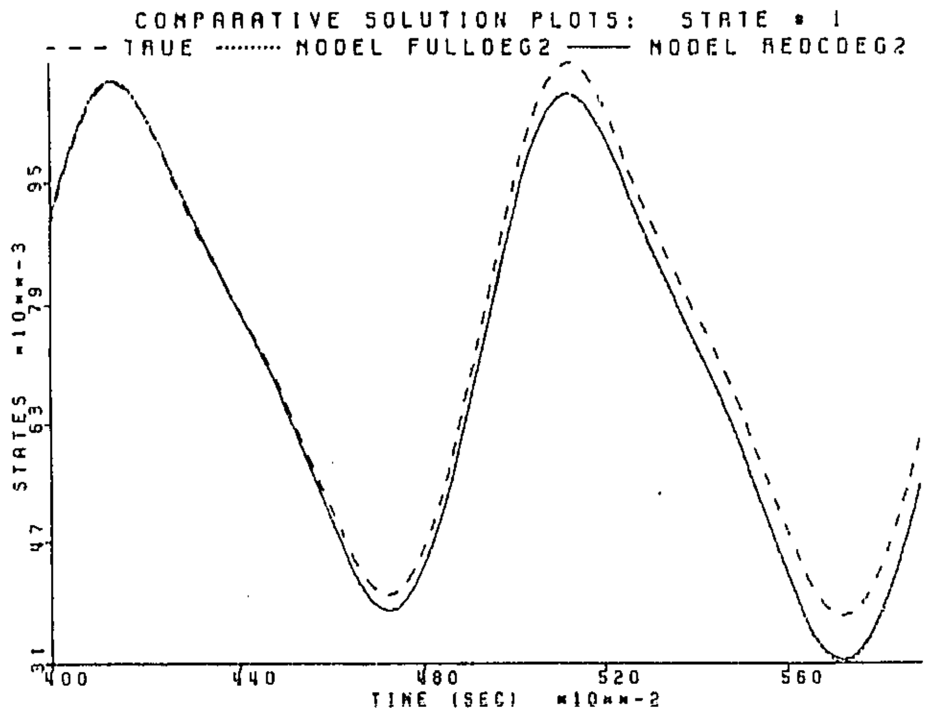



Figure 5.1.14 Simulation Number 19 of Table 5.1.10



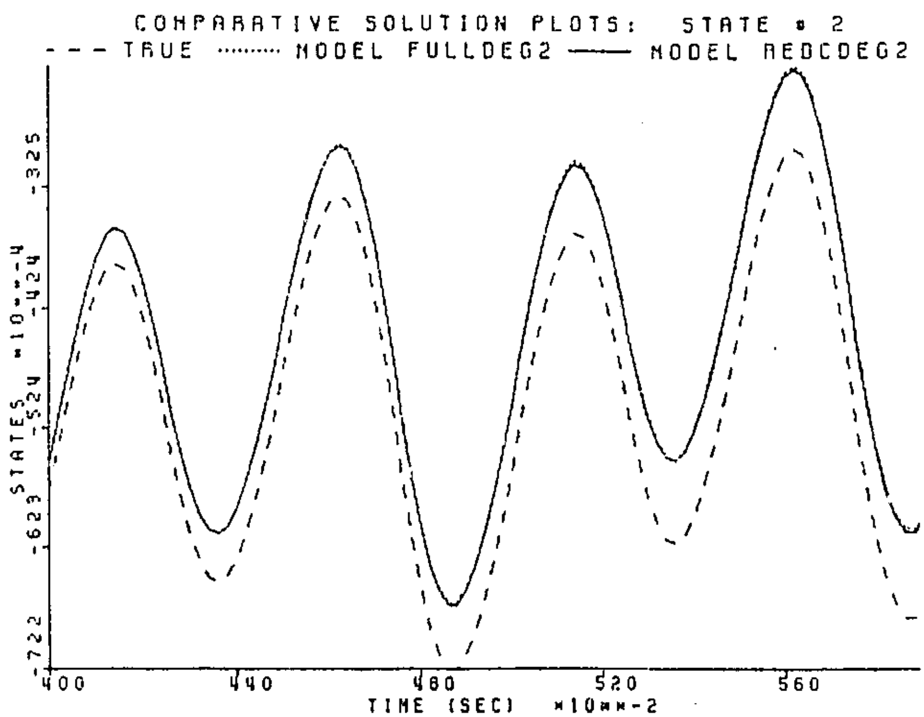
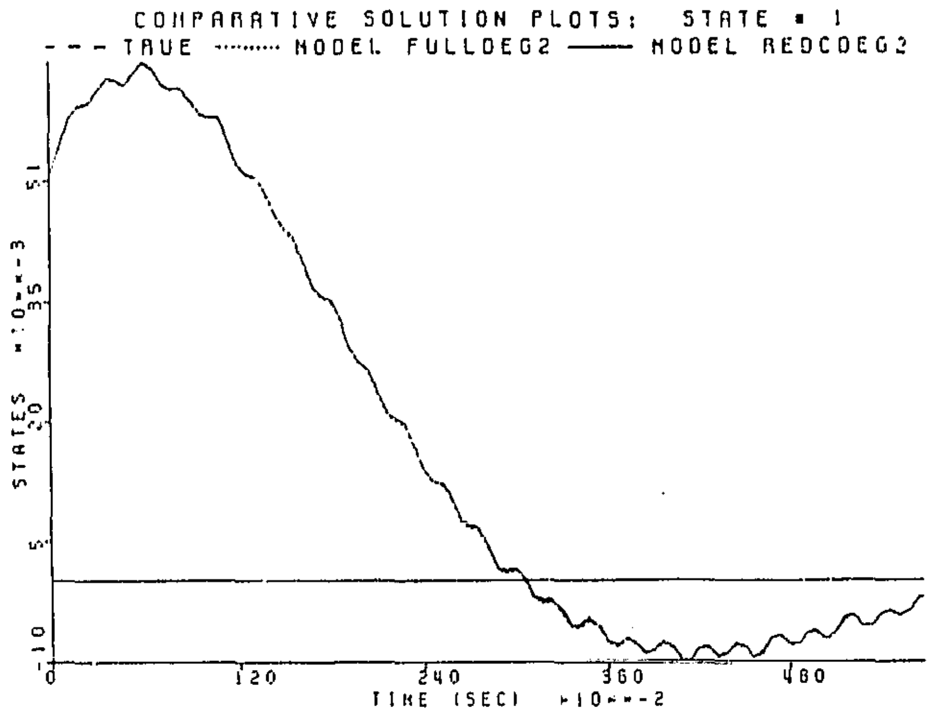




Figure 5.1.15 Expanded Section of Figure 5.1.14

```
****************************************
            PROBLEM SUMMARY
   CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
       DEGREE OF APPROXIMATION: 2
       # OF TERMS IN MODEL 2:   6
       DEGREE OF APPROXIMATION: 2
         SIMULATION WITH COSINE
****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.050 | -0.050 | -0.032 | -0.049 | 5.26 | 3.68 | 0.140 | 0.120 |
| 2 | 0.050 | -0.050 | 0.004 | 0.046 | 2.40 | 4.77 | 0.320E-01 | 0.898E-02 |
| 3 | 0.050 | -0.050 | -0.049 | 0.018 | 4.96 | 3.33 | 0.325 | 0.315 |
| 4 | 0.050 | -0.050 | 0.043 | -0.045 | 3.45 | 2.92 | 0.238 | 0.224 |
| 5 | 0.050 | -0.050 | -0.034 | -0.019 | 4.41 | 3.60 | 0.165 | 0.146 |
| 6 | 0.050 | -0.050 | 0.027 | -0.021 | 2.51 | 3.71 | 0.104 | 0.818E-01 |
| 7 | 0.050 | -0.050 | 0.020 | 0.028 | 5.60 | 2.08 | 0.647E-01 | 0.409E-01 |
| 8 | 0.050 | -0.050 | -0.027 | -0.033 | 5.71 | 4.73 | 0.104 | 0.819E-01 |
| 9 | 0.050 | -0.050 | -0.017 | -0.019 | 5.59 | 3.19 | 0.527E-01 | 0.284E-01 |
| 10 | 0.050 | -0.050 | 0.024 | 0.015 | 2.76 | 4.47 | 0.887E-01 | 0.657E-01 |
| 11 | 0.050 | -0.050 | 0.027 | -0.022 | 2.53 | 2.19 | 0.924E-01 | 0.766E-01 |
| 12 | 0.050 | -0.050 | 0.010 | 0.011 | 3.50 | 3.39 | 0.374E-01 | 0.103E-01 |
| 13 | 0.050 | -0.050 | 0.027 | -0.025 | 2.38 | 4.15 | 0.103 | 0.801E-01 |
| 14 | 0.050 | -0.050 | -0.041 | 0.024 | 3.52 | 2.87 | 0.227 | 0.211 |
| 15 | 0.050 | -0.050 | 0.045 | 0.049 | 3.51 | 2.91 | 0.279 | 0.267 |
| 16 | 0.050 | -0.050 | -0.006 | 0.046 | 3.72 | 5.72 | 0.325E-01 | 0.934E-02 |
| 17 | 0.050 | -0.050 | -0.006 | 0.026 | 5.79 | 5.32 | 0.322E-01 | 0.898E-02 |
| 18 | 0.050 | -0.050 | -0.025 | -0.027 | 4.14 | 5.40 | 0.942E-01 | 0.717E-01 |
| 19 | 0.050 | -0.050 | -0.021 | -0.010 | 2.01 | 4.65 | 0.671E-01 | 0.432E-01 |
| 20 | 0.050 | -0.050 | -0.043 | -0.017 | 3.74 | 2.51 | 0.257 | 0.243 |
| 21 | 0.050 | -0.050 | -0.011 | -0.022 | 2.57 | 3.82 | 0.387E-01 | 0.145E-01 |
| 22 | 0.050 | -0.050 | 0.037 | 0.035 | 3.59 | 4.53 | 0.195 | 0.179 |
| 23 | 0.050 | -0.050 | 0.002 | 0.002 | 3.50 | 2.78 | 0.327E-01 | 0.890E-02 |
| 24 | 0.050 | -0.050 | 0.028 | -0.000 | 2.30 | 3.67 | 0.113 | 0.907E-01 |
| 25 | 0.050 | -0.050 | 0.017 | -0.044 | 3.94 | 4.38 | 0.463E-01 | 0.258E-01 |
| 26 | 0.050 | -0.050 | 0.011 | 0.041 | 4.97 | 2.41 | 0.378E-01 | 0.140E-01 |
| 27 | 0.050 | -0.050 | -0.011 | -0.036 | 2.82 | 3.91 | 0.385E-01 | 0.147E-01 |
| 28 | 0.050 | -0.050 | -0.044 | -0.040 | 2.59 | 4.69 | 0.258 | 0.244 |
| 29 | 0.050 | -0.050 | 0.020 | 0.030 | 5.93 | 2.33 | 0.626E-01 | 0.387E-01 |
| 30 | 0.050 | -0.050 | -0.012 | 0.010 | 5.89 | 4.09 | 0.404E-01 | 0.160E-01 |
| 31 | 0.050 | -0.050 | 0.019 | -0.006 | 2.21 | 5.51 | 0.608E-01 | 0.365E-01 |
| 32 | 0.050 | -0.050 | -0.037 | 0.021 | 4.14 | 5.40 | 0.215 | 0.199 |
| 33 | 0.050 | -0.050 | -0.038 | -0.008 | 3.60 | 4.82 | 0.205 | 0.188 |
| 34 | 0.050 | -0.050 | 0.040 | -0.026 | 3.98 | 4.82 | 0.224 | 0.207 |
| 35 | 0.050 | -0.050 | -0.002 | -0.009 | 3.32 | 5.85 | 0.325E-01 | 0.875E-02 |
| 36 | 0.050 | -0.050 | -0.044 | 0.026 | 4.93 | 2.49 | 0.264 | 0.250 |
| 37 | 0.050 | -0.050 | -0.041 | -0.015 | 3.00 | 2.84 | 0.230 | 0.204 |
| 38 | 0.050 | -0.050 | -0.023 | 0.021 | 3.49 | 2.71 | 0.770E-01 | 0.533E-01 |
| 39 | 0.050 | -0.050 | 0.007 | -0.038 | 3.21 | 3.53 | 0.298E-01 | 0.968E-02 |
| 40 | 0.050 | -0.050 | 0.045 | 0.000 | 4.74 | 3.60 | 0.276 | 0.263 |
| 41 | 0.050 | -0.050 | 0.046 | -0.007 | 3.70 | 4.53 | 0.293 | 0.280 |
| 42 | 0.050 | -0.050 | 0.017 | 0.019 | 2.83 | 3.44 | 0.538E-01 | 0.297E-01 |
| 43 | 0.050 | -0.050 | 0.045 | 0.030 | 4.78 | 2.79 | 0.278 | 0.265 |
| 44 | 0.050 | -0.050 | 0.009 | -0.017 | 5.55 | 5.42 | 0.370E-01 | 0.161E-01 |
| 45 | 0.050 | -0.050 | -0.020 | -0.047 | 5.28 | 3.08 | 0.633E-01 | 0.396E-01 |
| 46 | 0.050 | -0.050 | -0.038 | 0.048 | 4.37 | 4.33 | 0.321 | 0.308 |
| 47 | 0.050 | -0.050 | 0.033 | -0.009 | 2.36 | 3.31 | 0.153 | 0.134 |
| 48 | 0.050 | -0.050 | 0.015 | -0.007 | 5.22 | 4.14 | 0.486E-01 | 0.240E-01 |
| 49 | 0.050 | -0.050 | -0.033 | 0.020 | 5.43 | 2.44 | 0.156 | 0.136 |
| 50 | 0.050 | -0.050 | -0.028 | 0.029 | 3.16 | 3.97 | 0.111 | 0.869E-01 |

Table 5.1.11 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



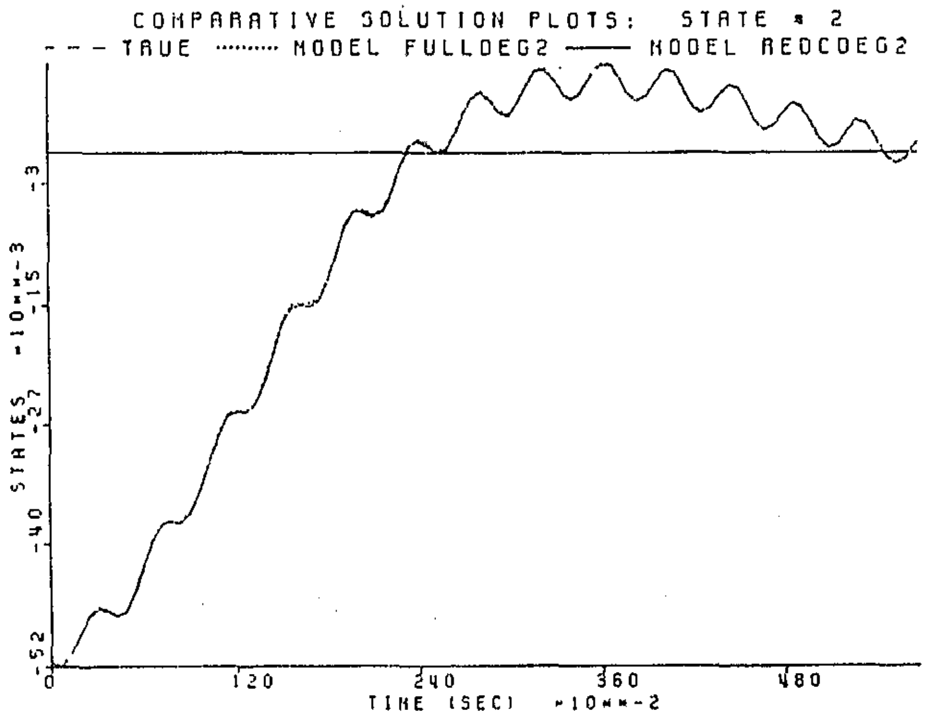




Figure 5.1.16 Simulation Number 13 of Table 5.1.11

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.075 | 0.075 | -0.249 | 0.243 | 4.40 | 2.12 | 1.03 | 0.983 |
| 2 | 0.075 | 0.075 | -0.041 | 0.179 | 5.70 | 5.30 | 2.45 | 3.91 |
| 3 | 0.075 | 0.075 | -0.147 | 0.192 | 3.09 | 3.27 | 1.69 | 1.61 |
| 4 | 0.075 | 0.075 | -0.205 | -0.164 | 2.82 | 3.50 | 1.13 | 1.11 |
| 5 | 0.075 | 0.075 | -0.090 | 0.082 | 3.86 | 5.54 | 2.03 | 2.22 |
| 6 | 0.075 | 0.075 | -0.171 | 0.070 | 3.41 | 5.45 | 1.23 | 1.24 |
| 7 | 0.075 | 0.075 | -0.155 | 0.162 | 4.59 | 2.50 | 1.34 | 1.36 |
| 8 | 0.075 | 0.075 | -0.108 | -0.171 | 2.41 | 2.67 | 1.62 | 1.63 |
| 9 | 0.075 | 0.075 | 0.017 | 0.066 | 5.82 | 2.42 | 3.33 | 6.46 |
| 10 | 0.075 | 0.075 | -0.186 | 0.180 | 4.91 | 5.17 | 1.29 | 1.27 |
| 11 | 0.075 | 0.075 | 0.140 | 0.188 | 3.55 | 4.25 | 1.46 | 1.49 |
| 12 | 0.075 | 0.075 | -0.035 | -0.145 | 2.02 | 4.44 | 3.36 | 5.78 |
| 13 | 0.075 | 0.075 | -0.055 | -0.145 | 5.69 | 4.90 | 3.01 | 4.06 |
| 14 | 0.075 | 0.075 | -0.024 | 0.070 | 4.27 | 2.63 | 3.31 | 6.05 |
| 15 | 0.075 | 0.075 | -0.114 | 0.154 | 3.72 | 5.06 | 1.70 | 1.79 |
| 16 | 0.075 | 0.075 | -0.085 | 0.086 | 4.57 | 3.16 | 2.13 | 2.37 |
| 17 | 0.075 | 0.075 | 0.129 | -0.134 | 3.34 | 3.57 | 1.75 | 1.84 |
| 18 | 0.075 | 0.075 | 0.120 | -0.122 | 3.43 | 5.18 | 1.61 | 1.67 |
| 19 | 0.075 | 0.075 | -0.069 | 0.096 | 5.64 | 4.58 | 2.52 | 3.03 |
| 20 | 0.075 | 0.075 | 0.217 | 0.223 | 5.88 | 2.14 | 1.11 | 1.07 |
| 21 | 0.075 | 0.075 | -0.006 | -0.234 | 5.06 | 3.57 | 3.58 | 8.91 |
| 22 | 0.075 | 0.075 | -0.175 | 0.074 | 4.13 | 3.97 | 1.24 | 1.22 |
| 23 | 0.075 | 0.075 | 0.160 | 0.179 | 2.63 | 4.95 | 1.33 | 1.34 |
| 24 | 0.075 | 0.075 | -0.240 | 0.035 | 5.84 | 2.90 | 1.03 | 0.997 |
| 25 | 0.075 | 0.075 | -0.117 | -0.124 | 4.33 | 2.47 | 1.65 | 1.72 |
| 26 | 0.075 | 0.075 | -0.179 | -0.058 | 4.63 | 4.11 | 1.20 | 1.20 |
| 27 | 0.075 | 0.075 | -0.091 | -0.241 | 4.40 | 5.94 | 2.23 | 2.52 |
| 28 | 0.075 | 0.075 | 0.184 | 0.223 | 3.83 | 4.44 | 1.22 | 1.20 |
| 29 | 0.075 | 0.075 | -0.056 | 0.249 | 5.96 | 5.37 | 2.84 | 4.15 |
| 30 | 0.075 | 0.075 | 0.225 | 0.106 | 3.39 | 4.23 | 1.07 | 1.04 |
| 31 | 0.075 | 0.075 | -0.054 | 0.023 | 2.73 | 2.81 | 2.93 | 3.66 |
| 32 | 0.075 | 0.075 | 0.036 | -0.195 | 3.44 | 3.88 | 2.65 | 4.45 |
| 33 | 0.075 | 0.075 | -0.034 | 0.220 | 2.20 | 5.13 | 3.52 | 6.72 |
| 34 | 0.075 | 0.075 | -0.239 | -0.235 | 3.23 | 2.52 | 1.06 | 1.01 |
| 35 | 0.075 | 0.075 | -0.182 | -0.139 | 5.47 | 2.94 | 1.20 | 1.20 |
| 36 | 0.075 | 0.075 | -0.129 | -0.062 | 4.81 | 3.67 | 1.50 | 1.54 |
| 37 | 0.075 | 0.075 | 0.131 | 0.013 | 4.82 | 4.70 | 1.47 | 1.52 |
| 38 | 0.075 | 0.075 | -0.163 | 0.170 | 5.15 | 2.38 | 1.29 | 1.31 |
| 39 | 0.075 | 0.075 | 0.158 | -0.186 | 5.27 | 2.75 | 1.33 | 1.35 |
| 40 | 0.075 | 0.075 | -0.031 | -0.045 | 5.50 | 2.12 | 3.26 | 5.50 |
| 41 | 0.075 | 0.075 | -0.193 | 0.187 | 3.34 | 5.53 | 1.17 | 1.16 |
| 42 | 0.075 | 0.075 | -0.037 | -0.024 | 3.35 | 4.16 | 3.20 | 5.06 |
| 43 | 0.075 | 0.075 | 0.220 | -0.098 | 3.94 | 4.60 | 1.08 | 1.05 |
| 44 | 0.075 | 0.075 | 0.144 | -0.205 | 3.16 | 3.83 | 1.41 | 1.46 |
| 45 | 0.075 | 0.075 | 0.157 | 0.196 | 3.77 | 4.01 | 1.22 | 1.24 |
| 46 | 0.075 | 0.075 | 0.126 | 0.131 | 4.21 | 2.40 | 1.55 | 1.60 |
| 47 | 0.075 | 0.075 | -0.226 | 0.020 | 2.30 | 5.98 | 1.07 | 1.04 |
| 48 | 0.075 | 0.075 | -0.170 | -0.087 | 5.25 | 4.87 | 1.25 | 1.25 |
| 49 | 0.075 | 0.075 | 0.216 | 0.148 | 2.21 | 5.99 | 1.10 | 1.07 |
| 50 | 0.075 | 0.075 | -0.197 | -0.218 | 4.06 | 2.30 | 1.17 | 1.15 |

Table 5.1.12 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

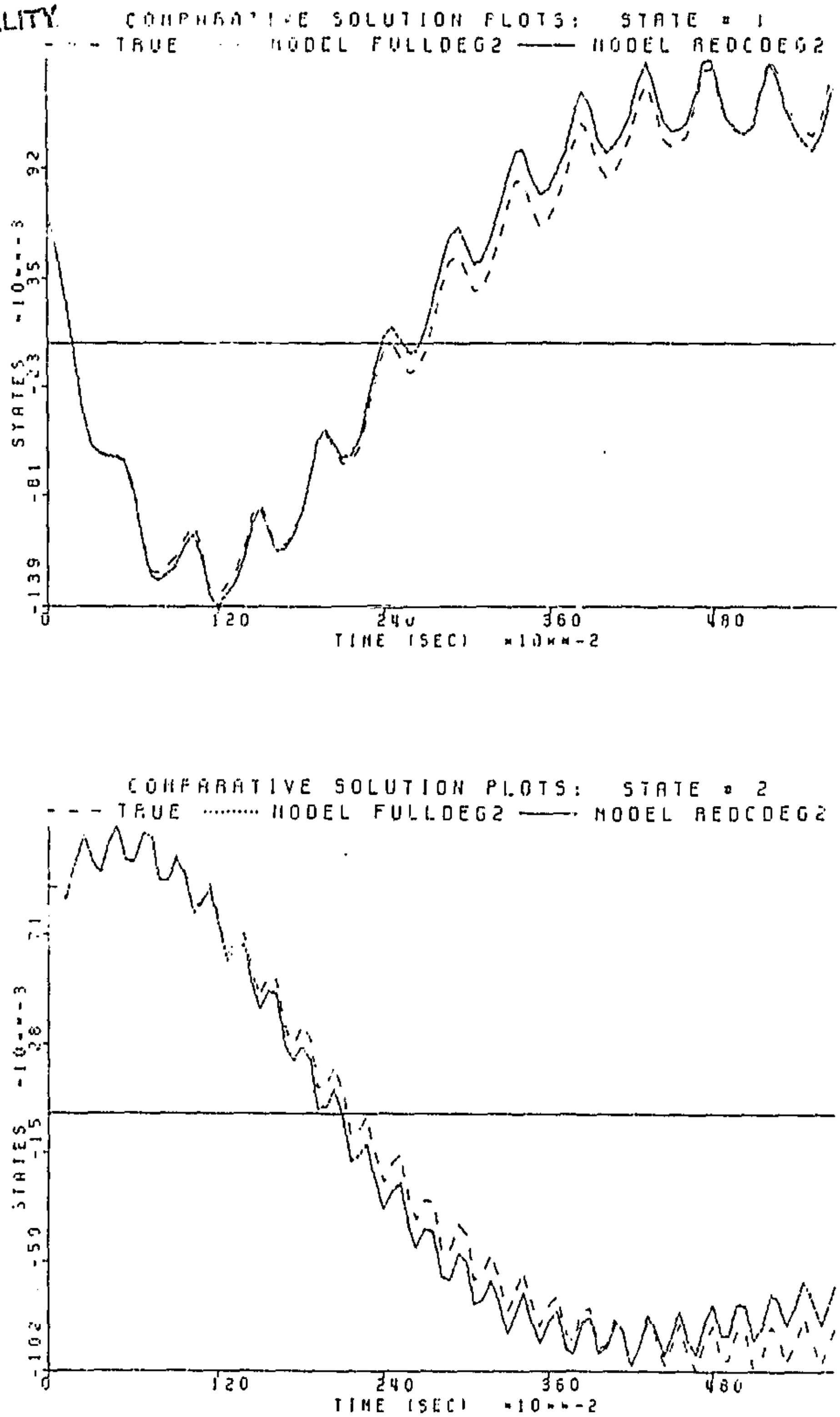
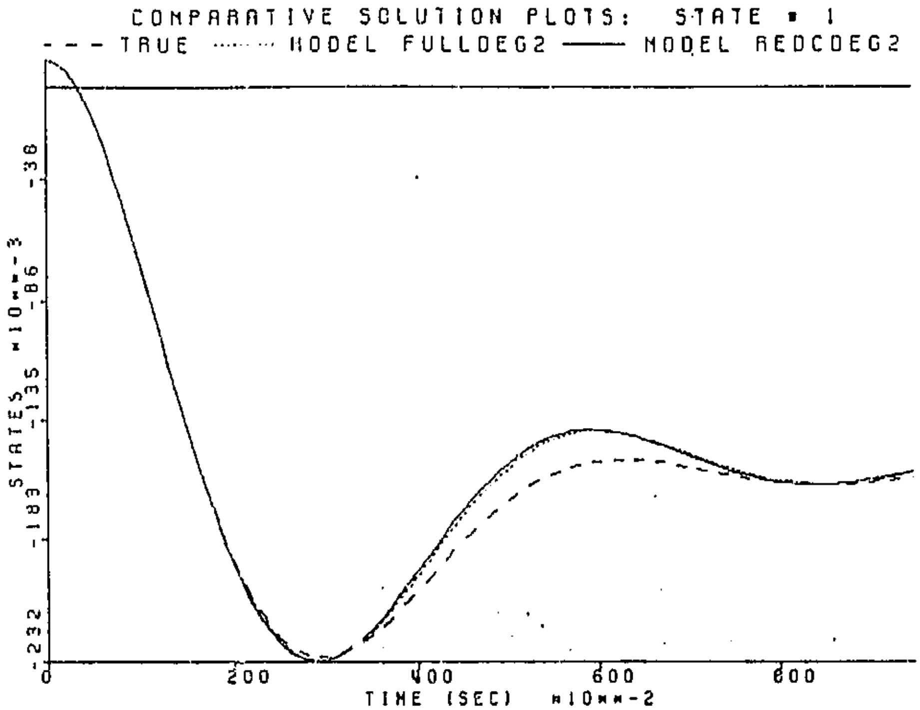



Figure 5.1.17 Simulation Number 1 of Table 5.1.1

*************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
*************************************

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | -0.001 | 0.010 | -0.075 | 0.00 | 0.00 | 0.312 | 0.318 |
| 2 | 0.001 | -0.001 | 0.010 | -0.075 | 0.01 | 0.01 | 0.300 | 0.307 |
| 3 | 0.001 | -0.001 | 0.010 | -0.075 | 0.02 | 0.02 | 0.260 | 0.274 |
| 4 | 0.001 | -0.001 | 0.010 | -0.075 | 0.05 | 0.05 | 0.210 | 0.232 |
| 5 | 0.001 | -0.001 | -0.050 | 0.050 | 0.00 | 0.00 | 1.67 | 1.74 |
| 6 | 0.001 | -0.001 | -0.050 | 0.050 | 0.01 | 0.01 | 1.73 | 1.80 |
| 7 | 0.001 | -0.001 | -0.050 | 0.050 | 0.02 | 0.02 | 1.99 | 2.03 |
| 8 | 0.001 | -0.001 | -0.050 | 0.050 | 0.05 | 0.05 | 3.13 | 1.52 |
| 9 | 0.001 | -0.001 | -0.075 | 0.010 | 0.00 | 0.00 | 1.20 | 1.23 |
| 10 | 0.001 | -0.001 | -0.075 | 0.010 | 0.01 | 0.01 | 1.23 | 1.26 |
| 11 | 0.001 | -0.001 | -0.075 | 0.010 | 0.02 | 0.02 | 1.34 | 1.35 |
| 12 | 0.001 | -0.001 | -0.075 | 0.010 | 0.05 | 0.05 | 1.81 | 1.07 |
| 13 | 0.001 | -0.001 | -0.075 | -0.075 | 0.00 | 0.00 | 0.694 | 0.682 |
| 14 | 0.001 | -0.001 | -0.075 | -0.075 | 0.01 | 0.01 | 0.713 | 0.695 |
| 15 | 0.001 | -0.001 | -0.075 | -0.075 | 0.02 | 0.02 | 0.783 | 0.754 |
| 16 | 0.001 | -0.001 | -0.075 | -0.075 | 0.05 | 0.05 | 0.907 | 0.774 |
| 17 | 0.010 | -0.010 | 0.010 | -0.075 | 0.00 | 0.00 | 0.310 | 0.316 |
| 18 | 0.010 | -0.010 | 0.010 | -0.075 | 0.01 | 0.01 | 0.298 | 0.305 |
| 19 | 0.010 | -0.010 | 0.010 | -0.075 | 0.02 | 0.02 | 0.258 | 0.271 |
| 20 | 0.010 | -0.010 | 0.010 | -0.075 | 0.05 | 0.05 | 0.208 | 0.229 |
| 21 | 0.010 | -0.010 | -0.050 | 0.050 | 0.00 | 0.00 | 1.67 | 1.73 |
| 22 | 0.010 | -0.010 | -0.050 | 0.050 | 0.01 | 0.01 | 1.72 | 1.79 |
| 23 | 0.010 | -0.010 | -0.050 | 0.050 | 0.02 | 0.02 | 1.98 | 2.02 |
| 24 | 0.010 | -0.010 | -0.050 | 0.050 | 0.05 | 0.05 | 3.08 | 1.53 |
| 25 | 0.010 | -0.010 | -0.075 | 0.010 | 0.00 | 0.00 | 1.21 | 1.24 |
| 26 | 0.010 | -0.010 | -0.075 | 0.010 | 0.01 | 0.01 | 1.23 | 1.26 |
| 27 | 0.010 | -0.010 | -0.075 | 0.010 | 0.02 | 0.02 | 1.34 | 1.36 |
| 28 | 0.010 | -0.010 | -0.075 | 0.010 | 0.05 | 0.05 | 1.83 | 1.08 |
| 29 | 0.010 | -0.010 | -0.075 | -0.075 | 0.00 | 0.00 | 0.697 | 0.683 |
| 30 | 0.010 | -0.010 | -0.075 | -0.075 | 0.01 | 0.01 | 0.716 | 0.696 |
| 31 | 0.010 | -0.010 | -0.075 | -0.075 | 0.02 | 0.02 | 0.787 | 0.756 |
| 32 | 0.010 | -0.010 | -0.075 | -0.075 | 0.05 | 0.05 | 0.915 | 0.772 |

Table 5.1.13 Low Frequency Table for Second Degree Full Model versus Second Degree Reduced Model



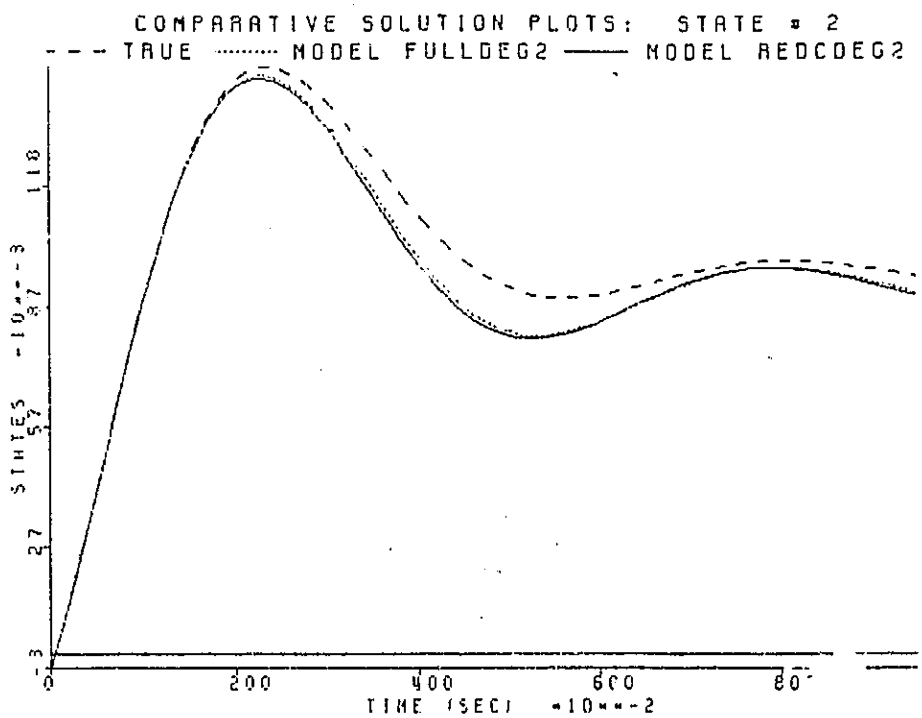




Figure 5.1.18 Simulation Number 26 of Table 5.1.13

were met. But, in this case the full model has only fourteen terms which is not unreasonably large. The true usefulness of the reduction method will appear when used on models with many nonlinear terms. With this in mind, the third degree approximation of this same two state system is studied.

### 5.1.2 DEGREE THREE APPROXIMATION

In this example, the third degree approximation now has 34 terms corresponding to the products

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1x_2,\ x_2x_2,$$
$$x_1u_1,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_1,\ u_1u_2,$$
$$u_2u_2,\ x_1x_1x_1,\ x_1x_1x_2,\ x_1x_2x_2,$$
$$x_2x_2x_2,\ x_1x_1u_1,\ x_1x_1u_2,\ x_1x_2u_1,\ x_1x_2u_2,$$
$$x_2x_2u_1,\ x_2x_2u_2,\ x_1u_1u_1,\ x_1u_1u_2,\ x_1u_2u_2,$$
$$x_2u_1u_1,\ x_2u_1u_2,\ x_2u_2u_2,\ u_1u_1u_1,\ u_1u_1u_2,$$
$$u_1u_2u_2,\ u_2u_2u_2.$$

The appendix contains the locally optimum, third degree model to be used as the starting model for the reduction. See Figure 5.1.19 for the original third degree model and the "first pass" reduction test. Note the model is identified using initial state conditions (0.001, -0.001). It also uses as control input excitations the same sum of two sinusoids, although with different amplitudes and frequencies,

$$u_1(t) = 0.055\sin(2\pi t\cdot\phi_1) + 0.055\sin(2\pi t\cdot 2\phi_1)$$
$$u_2(t) = 0.0325\sin(2\pi t\cdot\phi_2) + 0.0325\sin(2\pi t\cdot 2\phi_2)$$

where $\phi_1 = 0.5090$ and $\phi_2 = 2.0098$.

Directing our attention to the reduction test and the terms with changes in errors of magnitude less than 4, it is found that 18 terms can be omitted. The remaining 16 terms are

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010     -0.0010

ENTER SAMPLING PARAMETERS:
           APPROXIMATION DEGREE
           # SAMPLE POINTS
           INTERVAL BETWEEN SAMPLES
           INTEGRATION STEPSIZE
   3  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
           2            2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.5090      2.0098

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N


ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
           1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  34 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:         0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



TSO SPEAKEASY III PI+  9:57 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
```

Figure 5.1.19a  Full Third Degree Model and Reduction Test (First Pass)

```
-1.9977 -3.0008
 .99987  1.0002

...WITH EIGENVALUES:
 VALUES (A VECTOR WITH 2 COMPONENTS)
-.49874+.867671 -.49874-.867671

PARTITION NUMBER 2.
 PRTITION (A 2 BY 2 ARRAY)
 .0019891    .99779
-1.0001     -6.2231E-7

PARTITION NUMBER 3.
 PRTITION (A 2 BY 3 ARRAY)
-.19024      -.012365    -.0034771
 .036033      .0022184    1.5017E-4

PARTITION NUMBER 4.
 PRTITION (A 2 BY 4 ARRAY)
-4.0692      .026129     .027909    -.0078805
 .021083     .002591     .0026825    .0035398

PARTITION NUMBER 5.
 PRTITION (A 2 BY 3 ARRAY)
-.0039675    .0071601   -.012386
-.99787      .0010537    7.4681E-4

PARTITION NUMBER 6.
 PRTITION (A 2 BY 4 ARRAY)
 .70548  1.6627  3.3163  .78087
-.2223  -.6684  -1.2624 -.3395

PARTITION NUMBER 7.
 PRTITION (A 2 BY 6 ARRAY)
 .62148    -.61366    -.25223    -.76455     .44724    -.67544
-.28909    -.025264   -.05473    -.058202   -.048646   -.0056896

PARTITION NUMBER 8.
 PRTITION (A 2 BY 6 ARRAY)
-3.5488  -.10273   .41411    .16518    .036896   .075615
-.17155   .96002  -.044252 -.055923 -.035393 -.013022

PARTITION NUMBER 9.
 PRTITION (A 2 BY 4 ARRAY)
-.18738    -.032069  -.0251     -.019747
-.51749    -.016328  -.019048   -.0011802

 S (A 34 COMPONENT ARRAY)
 12.581    8.5987    7.997     6.0242    4.3769    4.1246    3.1931    2.9288
 2.6574    2.5454    2.1057    1.8416    1.5116    1.4722    1.3897    1.3138
 1.1668    .99995    .92334    .82698    .76732    .71888    .5978     .53451
 .40492    .35004    .32218    .20171    .18144    .11826    .023934   .045849
 .094429   .063016
 THE MAXIMUM SINGULAR VALUE:
MAX =  12.581
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .023934

 ...AND THEIR RATIO:
RTIO =  525.67
```

Figure 5.1.19b Full Third Degree Model and Reduction Test (First Pass)

```
WANT TO TRY REDUCTION TEST?[Y/N]: Y

FULL SYSTEM ERROR:      208.5798
FULL STATE ERRORS:      152.0251        56.5547

TERM:          X1,  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     273.2820
CHANGE IN STATE ERRORS:     226.0011        47.2809

TERM:          X2,  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:     -18.8934
CHANGE IN STATE ERRORS:     -16.9980        -1.8954

TERM:          U1,  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:     -25.4317
CHANGE IN STATE ERRORS:       0.0450       -25.4768

TERM:          U2,  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:      14.0826
CHANGE IN STATE ERRORS:      14.0826        -0.0000

TERM:       X1,X1,  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:      12.2573
CHANGE IN STATE ERRORS:      11.6348         0.6225

TERM:       X1,X2,  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:      -0.9152
CHANGE IN STATE ERRORS:      -0.8697        -0.0455

TERM:       X2,X2,  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:       0.2715
CHANGE IN STATE ERRORS:       0.2671         0.0044

TERM:       X1,U1,  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:      48.0619
CHANGE IN STATE ERRORS:      48.0063         0.0556

TERM:       X1,U2,  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:      -0.0250
CHANGE IN STATE ERRORS:      -0.0373         0.0123

TERM:       X2,U1,  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:      -0.3615
CHANGE IN STATE ERRORS:      -0.3495        -0.0120

TERM:       X2,U2,  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:      -0.0061
CHANGE IN STATE ERRORS:      -0.0176         0.0114

TERM:       U1,U1,  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -30.3638
CHANGE IN STATE ERRORS:      -0.0348       -30.3290

TERM:       U1,U2,  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:      -0.0164
CHANGE IN STATE ERRORS:      -0.0045        -0.0119

TERM:       U2,U2,  COLUMN #:  14.
CHANGE IN SYSTEM ERROR:       0.3772
CHANGE IN STATE ERRORS:       0.3599         0.0173

TERM:    X1,X1,X1,  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:     -[illegible].9081
CHANGE IN STATE ERRORS:     -27.9561        -1.9520
```

Figure 5.1.19c  Full Third Degree Model and Reduction Test (First Pass)

```
TERM:   X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:     136.7215
CHANGE IN STATE ERRORS:     122.6539        14.0676


TERM:   X1.X2.X2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:     -77.4448
CHANGE IN STATE ERRORS:     -75.6628        -1.7820

TERM:   X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:      79.0131
CHANGE IN STATE ERRORS:      69.5179         9.4952

TERM:   X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:       7.9639
CHANGE IN STATE ERRORS:       7.8924         0.0715

TERM:   X1.X1.U2.  COLUMN #:  20.
CHANGE IN SYSTEM ERROR:       1.3980
CHANGE IN STATE ERRORS:       1.5089        -0.1108

TERM:   X1.X2.U1.  COLUMN #:  21.
CHANGE IN SYSTEM ERROR:       1.7702
CHANGE IN STATE ERRORS:       1.6826         0.0876

TERM:   X1.X2.U2.  COLUMN #:  22.
CHANGE IN SYSTEM ERROR:       2.6437
CHANGE IN STATE ERRORS.       2.5047         0.1390

TERM:   X2.X2.U1.  COLUMN #:  23.
CHANGE IN SYSTEM ERROR:       2.3302
CHANGE IN STATE ERRORS:       2.7091        -0.3789

TERM:   X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:       5.6993
CHANGE IN STATE ERRORS:       5.7065        -0.0071

TERM:   X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:      57.3052
CHANGE IN STATE ERRORS:      58.0436        -0.7384

TERM:   X1.U1.U2.  COLUMN #:  26.
CHANGE IN SYSTEM ERROR:      -2.3094
CHANGE IN STATE ERRORS:       0.0114        -2.3208

TERM:   X1.U2.U2.  COLUMN #:  27.
CHANGE IN SYSTEM ERROR:      -8.9582
CHANGE IN STATE ERRORS:      -8.6700        -0.2882

TERM:   X2.U1.U1.  COLUMN #:  28.
CHANGE IN SYSTEM ERROR:       1.1869
CHANGE IN STATE ERRORS:       0.7543         0.4325

TERM:   X2.U1.U2.  COLUMN #:  29.
CHANGE IN SYSTEM ERROR:      -0.0037
CHANGE IN STATE ERRORS:       0.2052        -0.2090

TERM:   X2.U2.U2.  COLUMN #:  30.
CHANGE IN SYSTEM ERROR:       2.7262
CHANGE IN STATE ERRORS:       2.5737         0.1524

TERM:   U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:     -20.7670
CHANGE IN STATE ERRORS:      -2.2527       -18.5142

TERM:   U1.U1.U2.  COLUMN #:  32.
CHANGE IN SYSTEM ERROR:       0.2494
CHANGE IN STATE ERRORS:       0.2183         0.0311
```

Figure 5.1.19d  Full Third Degree Model and Reduction Test (First Pass)

```
TERM:   U1.U2.U2.  COLUMN !:  33.
CHANGE IN SYST"M ERROR:     -0.3957
CHANGE IN STATE ERRORS:     -0.2063       -0.1894

TERM:   U2.U2.U2.  COLUMN !:  34.
CHANGE IN SYSTEM ERROR:      0.2591
CHANGE IN STATE ERRORS:      0.2608       -0.0018

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 16

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(10:1,2,3,4,5,8,12,15,16,17)

A = ARRAY(6:18,19,24,25,27,31)

MANUAL MODE
 SPACE USED  113 K NOW,  138 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.19e  Full Third Degree Model and Reduction Test (First Pass)

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1u_1,\ u_1u_1,$$
$$x_1x_1x_1,\ x_1x_1x_2,\ x_1x_2x_2,\ x_2x_2x_2,$$
$$x_1x_1u_1,\ x_2x_2u_2,\ x_1u_1u_1,\ x_1u_2u_2,$$
$$u_1u_1u_1.$$

These terms are kept and a new local optimum is found. Optimization over frequencies yielded $\phi_1 = 0.3950$ and $\phi_2 = 1.3506$. See Figure 5.1.20 for the "second pass". Note that the IMSL optimization routine ended with a "terminal error". This error (IER = 130) is an IMSL provided error check to insure that roundoff error does not become excessive. Note further that a FORTRAN error message summary is provided. In this instance error 208 was encountered 34 times. Error 208 is an underflow error and indicates that an internal variable was less than $10^{-78}$. The standard fixup is to set that variable to zero, which is perfectly acceptable.

The "second pass" reduction test indicates that another 3 terms are fairly insignificant. This brings the total number of terms to 13. They are

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1u_1,\ u_1u_1,$$
$$x_1x_1x_1,\ x_1x_1x_2,\ x_2x_2x_2,\ x_1x_1u_1,$$
$$x_2x_2u_2,\ x_1u_1u_1,\ u_1u_1u_1.$$

Again, only these 13 terms are kept and a locally optimum frequency set is determined. The model is identified and a reduction test performed. (See Figure 5.1.21 for this "third pass".) The reduction test indicates that term 19 ($x_1x_1u_1$) could be removed, but after execution of the "fourth pass", the model simulations showed that the 12 term model was not acceptable. So the 13 term model is the final model, and the final reduction is 21 terms removed out of a possible 34; a 62% reduction.

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010     -0.0010

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   3  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2           2

ENTER THE   4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.5090      2.0098

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
           3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2    200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.5253147D+00  0.2009603D+01
THE CONDITION NUMBER:  0.152D+10
OBJECTIVE FUNCTION VALUE: 0.5532D+04

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.5254612D+00  0.2008699D+01
THE CONDITION NUMBER:  0.152D+10
OBJECTIVE FUNCTION VALUE: 0.5516D+04

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.5199821D+00  0.1989839D+01
THE CONDITION NUMBER:  0.145D+10
OBJECTIVE FUNCTION VALUE: 0.5316D+04
```

Figure 5.1.20a  Reduced Third Degree Model and Reduction Test (Second Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.5039734D+00  0.1890305D+01
THE CONDITION NUMBER:  0.123D+10
OBJECTIVE FUNCTION VALUE: 0.4721D+04

 50 CALLS OF FCN.....
THE PARAMETERS:
 0.4076332D+00  0.1414816D+01
THE CONDITION NUMBER:  0.470D+09
OBJECTIVE FUNCTION VALUE: 0.2133D+04
*** TERMINAL ERROR -         (IER = 130) FROM IMSL ROUTINE ZXMIN

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  16 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:       52

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.395     FREQ(2)=  1.351     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



 MESSAGE SUMMARY: MESSAGE NUMBER - COUNT
                               208      34

TSO SPEAKEASY III PI+ 10:59 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 16 COMPONENT ARRAY)
  1   2   3   4   5   8   12  15  16  17  18  19  24  25  27  31


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -1.9999 -3.0002
  1.0011  .99978

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50005+.86841i -.50005-.86841i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.5116E-4  .99902
 -.99844    -.0013877


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
 -.028226  0        0
 -.093347  0        0
```

Figure 5.1.20b  Reduced Third Degree Model and Reduction Test (Second Pass)

```
PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0367   0        0        0
 -.062668  0        0        0


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.0022205  0        0
 -1.0093    0        0


PARTITION NUMBER 6.
  PRTITION (A 2 BY 4 ARRAY)
 -.91373   .39023   .40494 -.40803
  .48991 -1.1593  -.11776   .55633


PARTITION NUMBER 7.
  PRTITION (A 2 BY 6 ARRAY)
  .11824   0        0        0        0        .049393
  .58555   0        0        0        0        .62134


PARTITION NUMBER 8.
  PRTITION (A 2 BY 6 ARRAY)
 -3.9208   0        .028512  0        0        0
  .48114   0        .12065   0        0        0


PARTITION NUMBER 9.
  PRTITION (A 2 BY 4 ARRAY)
 -.058225  0        0        0
 -.57913   0        0        0


  S (A 16 COMPONENT ARRAY)
  11.449    7.1423   6.0617   5.0361   2.7425   2.088    1.9426   1.3796
  1.1688    1.1436   .8219    .56891   .31614   .17437   .084745  .034909
 THE MAXIMUM SINGULAR VALUE:
MAX =   11.449
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =   .034909

 ...AND THEIR RATIO:
RTIO =   327.96


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     138.4610
FULL STATE ERRORS:      70.3943      68.0666

TERM:           X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:    105.5239
CHANGE IN STATE ERRORS:    109.8157      -4.2918

TERM:           X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:    234.7848
CHANGE IN STATE ERRORS:    194.0638      40.7210

TERM:           U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:    -41.7820
CHANGE IN STATE ERRORS:     -0.0060     -41.7761

TERM:           U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:    -16.5398
```

Figure 5.1.20c  Reduced Third Degree Model and Reduction Test (Second Pass)

```
CHANGE IN STATE ERRORS:     -16.5602         0.0204

TERM:       X1.X1.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:       3.0913
CHANGE IN STATE ERRORS:       0.0624         3.0289

TERM:       X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:     113.9193
CHANGE IN STATE ERRORS:     114.6507        -0.7314

TERM:       U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -19.9546
CHANGE IN STATE ERRORS:      -0.0270       -19.9276

TERM:    X1.X1.X1.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:       4.4767
CHANGE IN STATE ERRORS:      15.1272       -10.6505

TERM:    X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:      -2.5463
CHANGE IN STATE ERRORS:       5.6283        -8.1746

TERM:    X1.X2.X2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:      -1.4812
CHANGE IN STATE ERRORS:      -3.6453         2.1641

TERM:    X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:       2.7722
CHANGE IN STATE ERRORS:      -7.2330        10.0051

TERM:    X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:       7.1812
CHANGE IN STATE ERRORS:      -0.6173         7.7985

TERM:    X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:      -6.0369
CHANGE IN STATE ERRORS:      -0.9311        -5.1057

TERM:    X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:       9.4809
CHANGE IN STATE ERRORS:      11.9710        -2.4901

TERM:    X1.U2.U2.  COLUMN #:  27.
CHANGE IN SYSTEM ERROR:      -1.2075
CHANGE IN STATE ERRORS:       0.1620        -1.3695

TERM:    U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:     -24.7955
CHANGE IN STATE ERRORS:       0.7590       -25.5545

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 13

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(1,2,3,4,8,12,15,16,18,19)

A = ARRAY(24,25,31)

MANUAL MODE
 SPACE USED  70 K NOW,  81 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.20d  Reduced Third Degree Model and Reduction Test (Second Pass)

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010     -0.0010

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   3  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2             2

ENTER THE   4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.3950      1.3506

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2    200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.3986263D+00  0.1343124D+01
THE CONDITION NUMBER:  0.400D+08
OBJECTIVE FUNCTION VALUE: 0.1685D+03

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.3991838D+00  0.1352194D+01
THE CONDITION NUMBER:  0.397D+08
OBJECTIVE FUNCTION VALUE: 0.1681D+03

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.3991642D+00  0.1352155D+01
THE CONDITION NUMBER:  0.397D+08
OBJECTIVE FUNCTION VALUE: 0.1681D+03
```

Figure 5.1.21a  Reduced Third Degree Model and Reduction Test (Third Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.3991505D+00  0.1352110D+01
THE CONDITION NUMBER:  0.397D+03
OBJECTIVE FUNCTION VALUE: 0.1681D+03

CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  13 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        48 .

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.399      FREQ(2)=  1.352     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



 MESSAGE SUMMARY: MESSAGE NUMBER - COUNT
                            208       55

TSO SPEAKEASY III PI+ 11:30 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 13 COMPONENT ARRAY)
  1   2   3   4   8   12  15  16  18  19  24  25  31


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -1.9999 -2.9999
  .99957  .99945

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50021+.86582i -.50021-.86582i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.3827E-4   .99898
 -.99649    -.0014354


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  0  0  0
  0  0  0


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0226   0         0         0
 -.026954  0         0         0
```

Figure 5.1.21b  Reduced Third Degree Model and Reduction Test (Third Pass)

C-2

```
PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -6.5226E-4  0          0
 -1.0026     0          0


PARTITION NUMBER 6.
  PRTITION (A 2 BY 4 ARRAY)
 -1.2411   .15993  0        -.5407
 -.26768 -.91146   0         .77552


PARTITION NUMBER 7.
  PRTITION (A 2 BY 6 ARRAY)
 -.063216  0         0         0         0         .073678
  .20884   0         0         0         0         .68143


PARTITION NUMBER 8.
  PRTITION (A 2 BY 6 ARRAY)
 -4.0074  0        0        0        0        0
  .27416  0        0        0        0        0


PARTITION NUMBER 9.
  PRTITION (A 2 BY 4 ARRAY)
 -.078589  0         0         0
 -.66691   0         0         0


  S (A 13 COMPONENT ARRAY)
  9.8648  7.1115  6.0803  4.7378  2.4718  1.9875  1.3949  1.2823  1.1637
  .95869  .56438  .26112  .16092
 THE MAXIMUM SINGULAR VALUE:
MAX =  9.8648
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .16092

 ...AND THEIR RATIO:
RTIO =  61.303



WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     133.9531
FULL STATE ERRORS:      70.3439      63.6092

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:    121.0994
CHANGE IN STATE ERRORS:     85.4754      35.6240

TERM:          X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:    222.9544
CHANGE IN STATE ERRORS:    220.0404       2.9140

TERM:          U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:    -37.8873
CHANGE IN STATE ERRORS:     -0.0055     -37.8818

TERM:          U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:    -15.6390
CHANGE IN STATE ERRORS:    -15.6634       0.0243

TERM:       X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:    123.9281
```

Figure 5.1.21c  Reduced Third Degree Model and Reduction Test (Third Pass)

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```
CHANGE IN STATE ERRORS:     124.1291        -0.2010

TERM:       U1.U1.  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:    -27.5552
CHANGE IN STATE ERRORS:     -0.0115        -27.5438

TERM:    X1.X1.X1.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:     15.9879
CHANGE IN STATE ERRORS:     15.2055          0.7824

TERM:    X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:     16.6559
CHANGE IN STATE ERRORS:      0.5196         16.1362

TERM:    X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:     -8.7379
CHANGE IN STATE ERRORS:     -4.3799         -4.3580

TERM:    X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:      1.9149
CHANGE IN STATE ERRORS:      0.4854          1.4295

TERM:    X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:     -5.8982
CHANGE IN STATE ERRORS:     -1.3748         -4.5234

TERM:    X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:      7.7140
CHANGE IN STATE ERRORS:      7.7567         -0.0427

TERM:    U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:    -25.8595
CHANGE IN STATE ERRORS:      0.9593        -26.8188

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N

MANUAL MODE
 SPACE USED  62 K NOW,  71 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.21d  Reduced Third Degree Model and Reduction Test (Third Pass)

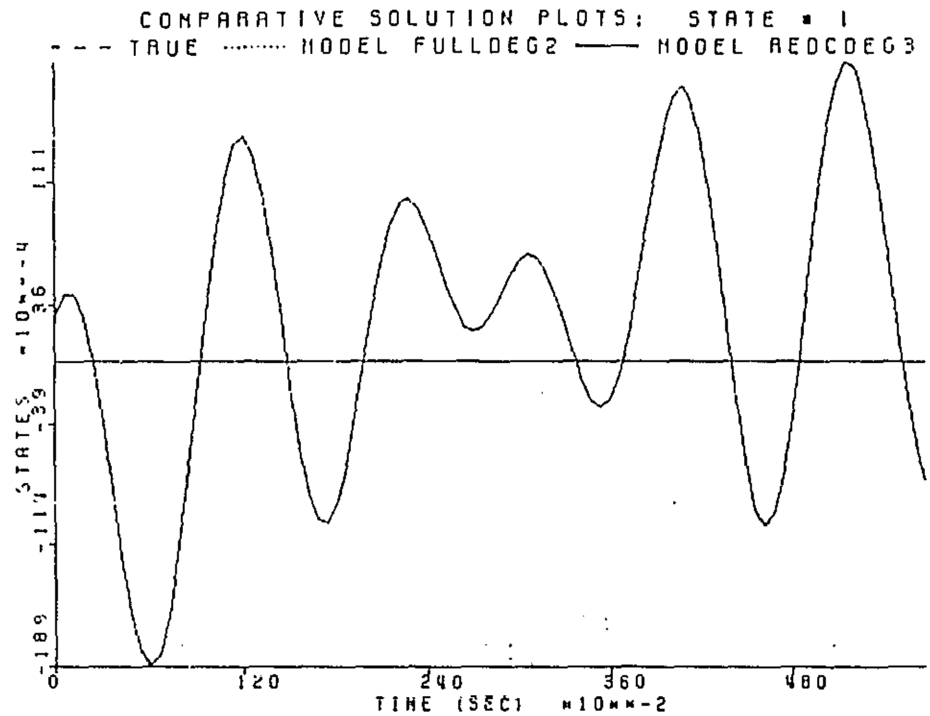
Now all that remains is simulation to bear out the model's validity. First we test against the full second degree model and the error ratio for each state is the mean square of the reduced third degree over the mean square error of the full second degree. In each of the following tables MODEL1 is the second degree model and MODEL2 is the reduced third degree. The first two tables (Table 5.1.14 and Table 5.1.15) and their corresponding plots display the behavior when excitation parameters are chosen close to the origin. The following three tables stretch the initial conditions and amplitudes farther from the origin. From this set of data, it is clear that the reduced third degree model outperforms the full second degree. Table 5.1.19 is a table including randomly chosen amplitudes and frequencies. The corresponding figures (Figures 5.1.29 and 5.1.30) show a worst case and a typical plot of model behavior. The final two tables show the low frequency and d.c. behavior of the two models. Again a definite improvement is observed.

Now, comparison against the full degree three model is in order. The error ratios for each state are now the mean square error of the reduced model over the mean square error of the full third degree approximation. The first table (Table 5.1.22) and its corresponding set of plots (Figure 5.1.34) show the typical comparison of model behavior near the origin. The next two tables move the control amplitudes farther out. Throughout these tables, the reduced model behavior is quite good and sometimes even better than that of the full model, but the reduced model went unstable as the amplitudes reached ±0.6, while the full model remained stable and did well (see the appendix). Table 5.1.25 is another random table with amplitudes in the range (-0.4, 0.4) and frequencies in the range (2,6). Table 5.1.26 is a low frequency table and Table 5.1.27 is a table of random steps. In these cases, we note that re-

```
************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2: 13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.148E-01 | 0.133E-01 |
| 2 | 0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.226 | 0.394 |
| 3 | 0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.197 | 0.419 |
| 4 | 0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.279 | 0.639 |
| 5 | 0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.614 | 1.21 |
| 6 | 0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.149 | 0.947E-01 |
| 7 | 0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.205 | 0.114 |
| 8 | 0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.141 | 0.879E-01 |
| 9 | 0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.181 | 0.105 |
| 10 | 0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.149E-01 | 0.107E-01 |
| 11 | 0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.321 | 0.599 |
| 12 | 0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.208 | 0.415 |
| 13 | 0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.385 | 0.886 |
| 14 | 0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.705 | 1.57 |
| 15 | 0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.144 | 0.929E-01 |
| 16 | 0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.185 | 0.108 |
| 17 | 0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.134 | 0.852E-01 |
| 18 | 0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.168 | 0.101 |
| 19 | -0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.187E-01 | 0.168E-01 |
| 20 | -0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.323 | 0.647 |
| 21 | -0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.238 | 0.472 |
| 22 | -0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.401 | 0.907 |
| 23 | -0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.850 | 1.90 |
| 24 | -0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.141 | 0.919E-01 |
| 25 | -0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.180 | 0.105 |
| 26 | -0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.132 | 0.844E-01 |
| 27 | -0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.161 | 0.992E-01 |
| 28 | -0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.131E-01 | 0.943E-02 |
| 29 | -0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.229 | 0.428 |
| 30 | -0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.222 | 0.475 |
| 31 | -0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.319 | 0.737 |
| 32 | -0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.753 | 1.42 |
| 33 | -0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 0.147 | 0.939E-01 |
| 34 | -0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 0.200 | 0.111 |
| 35 | -0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 0.139 | 0.871E-01 |
| 36 | -0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 0.173 | 0.103 |

Table 5.1.14 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model



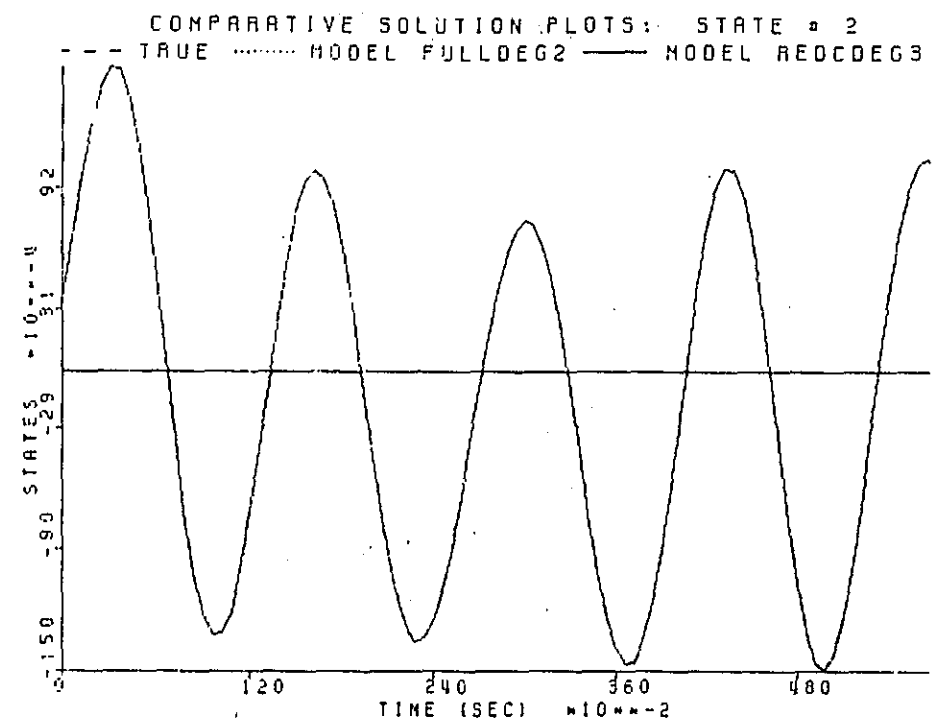




Figure 5.1.22 Simulation Number 5 of Table 5.1.14

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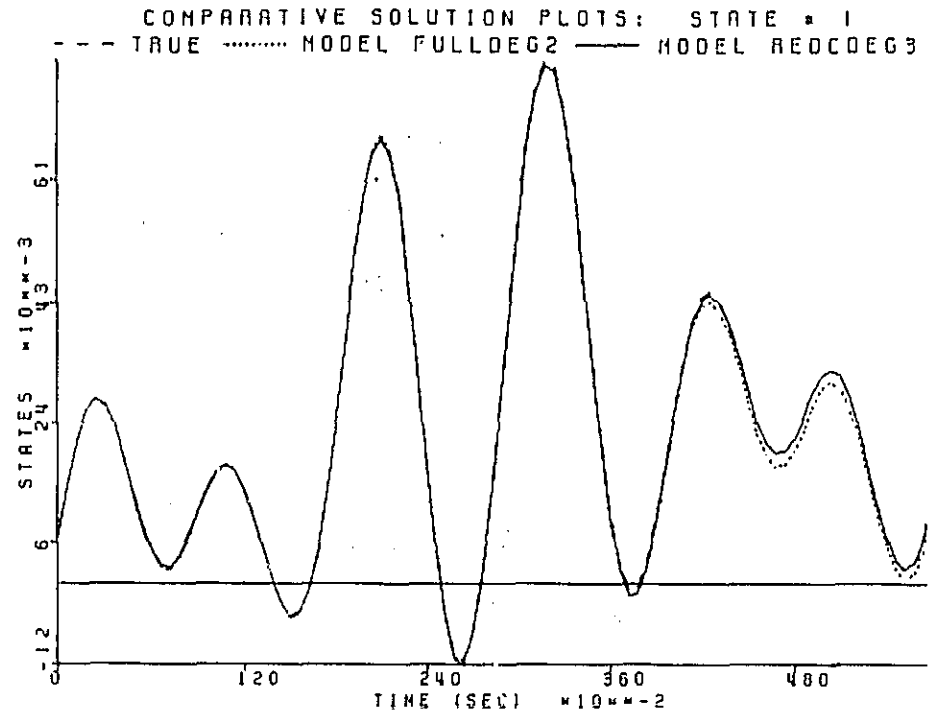


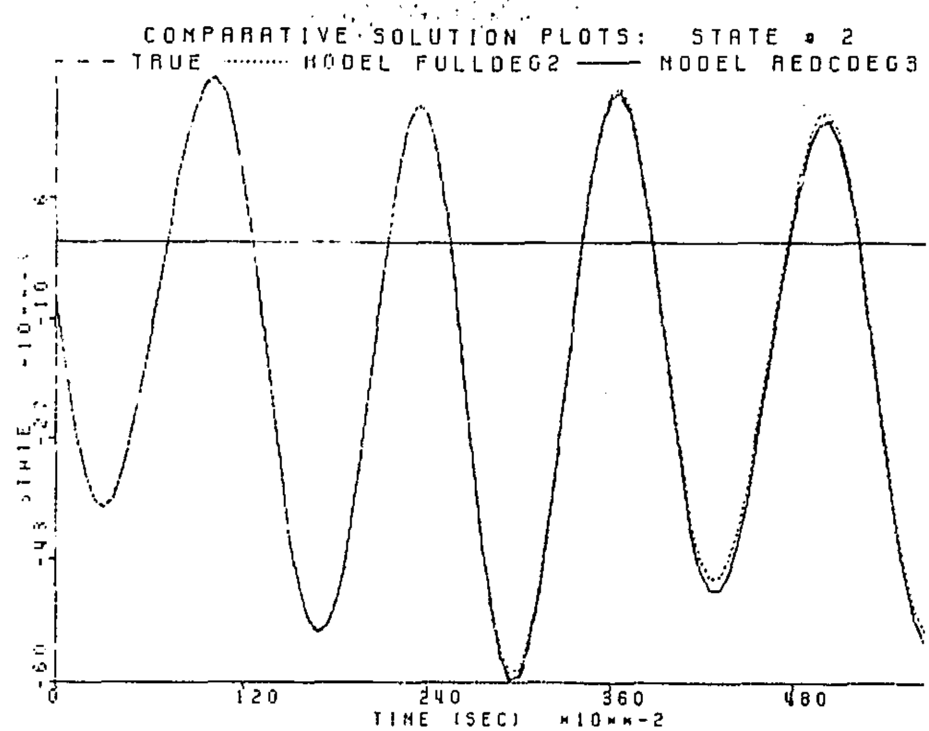
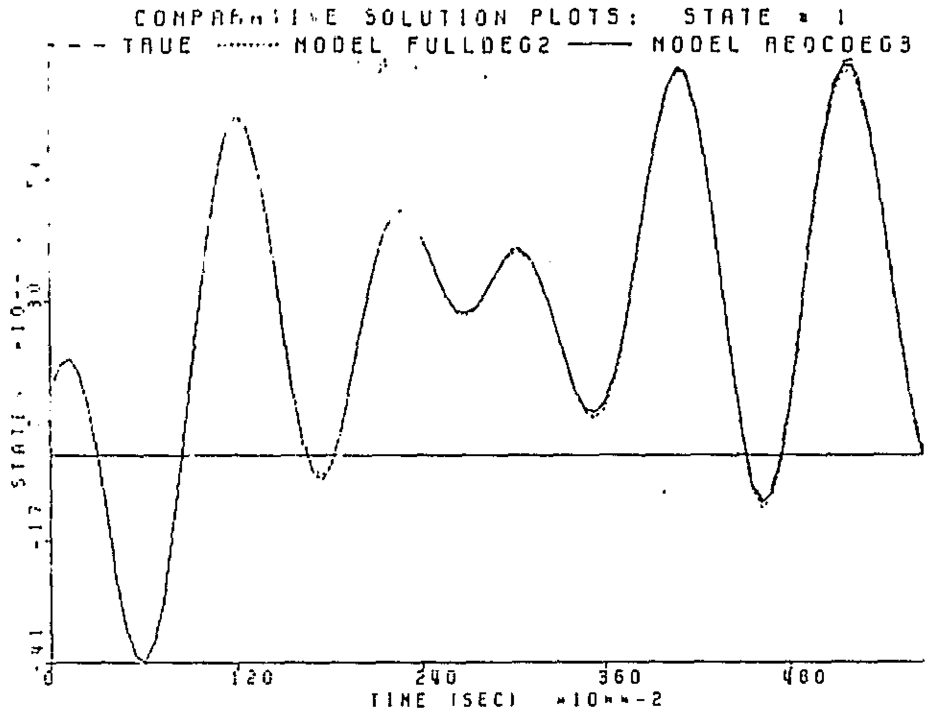




Figure 5.1.23 Simulation Number 33 of Table 5.1.14

```
****************************************
           PROBLEM SUMMARY
  CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:  14
      DEGREE OF APPROXIMATION: 2
      # OF TERMS IN MODEL 2:  13
      DEGREE OF APPROXIMATION: 3
        SIMULATION WITH COSINE
****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.005 | 0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.101E-01 | 0.899E-02 |
| 2 | 0.005 | 0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.164 | 0.221 |
| 3 | 0.005 | 0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.154 | 0.345 |
| 4 | 0.005 | 0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.966E-01 | 0.226 |
| 5 | 0.005 | 0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.344 | 0.681 |
| 6 | 0.005 | 0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.155 | 0.954E-01 |
| 7 | 0.005 | 0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.260 | 0.133 |
| 8 | 0.005 | 0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.153 | 0.929E-01 |
| 9 | 0.005 | 0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.227 | 0.118 |
| 10 | 0.005 | -0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.206E-01 | 0.148E-01 |
| 11 | 0.005 | -0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.564 | 1.19 |
| 12 | 0.005 | -0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.176 | 0.296 |
| 13 | 0.005 | -0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.446 | 0.980 |
| 14 | 0.005 | -0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.515 | 1.28 |
| 15 | 0.005 | -0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.136 | 0.878E-01 |
| 16 | 0.005 | -0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.158 | 0.100 |
| 17 | 0.005 | -0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.126 | 0.813E-01 |
| 18 | 0.005 | -0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.158 | 0.966E-01 |
| 19 | -0.005 | -0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.344E-01 | 0.306E-01 |
| 20 | -0.005 | -0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.316 | 0.595 |
| 21 | -0.005 | -0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.300 | 0.458 |
| 22 | -0.005 | -0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.272 | 0.415 |
| 23 | -0.005 | -0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.447 | 1.26 |
| 24 | -0.005 | -0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.124 | 0.849E-01 |
| 25 | -0.005 | -0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.139 | 0.895E-01 |
| 26 | -0.005 | -0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.116 | 0.770E-01 |
| 27 | -0.005 | -0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.128 | 0.876E-01 |
| 28 | -0.005 | 0.005 | 0.000 | 0.000 | 0.75 | 1.00 | 0.106E-01 | 0.761E-02 |
| 29 | -0.005 | 0.005 | 0.050 | 0.050 | 0.75 | 1.00 | 0.127 | 0.251 |
| 30 | -0.005 | 0.005 | 0.050 | -0.050 | 0.75 | 1.00 | 0.246 | 0.560 |
| 31 | -0.005 | 0.005 | -0.050 | -0.050 | 0.75 | 1.00 | 0.202 | 0.476 |
| 32 | -0.005 | 0.005 | -0.050 | 0.050 | 0.75 | 1.00 | 0.777 | 1.05 |
| 33 | -0.005 | 0.005 | 0.150 | 0.150 | 0.75 | 1.00 | 0.149 | 0.946E-01 |
| 34 | -0.005 | 0.005 | 0.150 | -0.150 | 0.75 | 1.00 | 0.234 | 0.117 |
| 35 | -0.005 | 0.005 | -0.150 | -0.150 | 0.75 | 1.00 | 0.146 | 0.907E-01 |
| 36 | -0.005 | 0.005 | -0.150 | 0.150 | 0.75 | 1.00 | 0.180 | 0.107 |

Table 5.1.15 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model



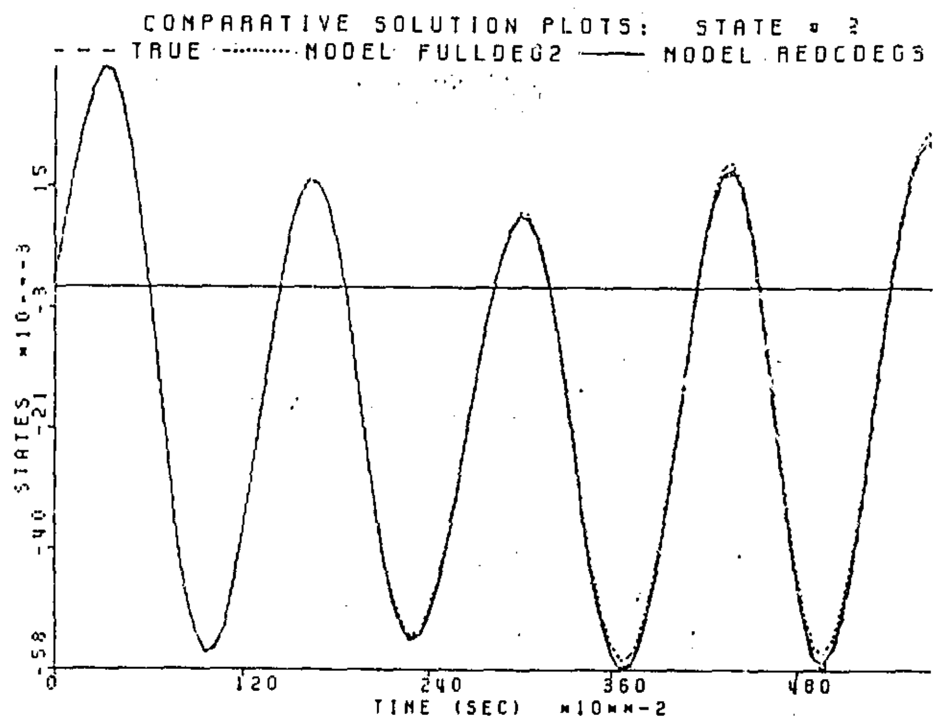




Figure 5.1.24 Simulation Number 18 of Table 5.1.15

```
**************************************
           PROBLEM SUMMARY
  CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:  14
      DEGREE OF APPROXIMATION: 2
      # OF TERMS IN MODEL 2:  13
      DEGREE OF APPROXIMATION: 3
        SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.518E-01 | 0.453E-01 |
| 2 | 0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.384E-01 | 0.351E-01 |
| 3 | 0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.181E-01 | 0.231E-01 |
| 4 | 0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.234E-01 | 0.260E-01 |
| 5 | 0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.143E-01 | 0.150E-01 |
| 6 | 0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.682E-02 | 0.101E-01 |
| 7 | 0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.576E-01 | 0.391E-01 |
| 8 | 0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.569E-01 | 0.394E-01 |
| 9 | -0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.391E-01 | 0.364E-01 |
| 10 | -0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.385E-01 | 0.341E-01 |
| 11 | -0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.432E-01 | 0.373E-01 |
| 12 | -0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.541E-01 | 0.437E-01 |
| 13 | -0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.836E-02 | 0.727E-02 |
| 14 | -0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.644E-02 | 0.644E-02 |
| 15 | -0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.217E-01 | 0.169E-01 |
| 16 | -0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.256E-01 | 0.187E-01 |
| 17 | 0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.476E-01 | 0.255E-01 |
| 18 | 0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.446E-01 | 0.201E-01 |
| 19 | 0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.143 | 0.584E-01 |
| 20 | 0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.179 | 0.659E-01 |
| 21 | 0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.348E-01 | 0.178E-01 |
| 22 | 0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.326E-01 | 0.166E-01 |
| 23 | 0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.900E-01 | 0.471E-01 |
| 24 | 0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.945E-01 | 0.501E-01 |
| 25 | -0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.363E-01 | 0.195E-01 |
| 26 | -0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.389E-01 | 0.181E-01 |
| 27 | -0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.109 | 0.506E-01 |
| 28 | -0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.123 | 0.550E-01 |
| 29 | -0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 0.252E-01 | 0.149E-01 |
| 30 | -0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 0.238E-01 | 0.143E-01 |
| 31 | -0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 0.567E-01 | 0.351E-01 |
| 32 | -0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 0.615E-01 | 0.377E-01 |
| 33 | 0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.552E-01 | 0.243E-01 |
| 34 | 0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.606E-01 | 0.241E-01 |
| 35 | 0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.126 | 0.545E-01 |
| 36 | 0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.137 | 0.588E-01 |
| 37 | 0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.467E-01 | 0.242E-01 |
| 38 | 0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.467E-01 | 0.238E-01 |
| 39 | 0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.907E-01 | 0.495E-01 |
| 40 | 0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.964E-01 | 0.531E-01 |
| 41 | -0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.467E-01 | 0.232E-01 |
| 42 | -0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.496E-01 | 0.231E-01 |
| 43 | -0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.999E-01 | 0.501E-01 |
| 44 | -0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.107 | 0.537E-01 |
| 45 | -0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 0.391E-01 | 0.226E-01 |
| 46 | -0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 0.385E-01 | 0.221E-01 |
| 47 | -0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 0.723E-01 | 0.443E-01 |
| 48 | -0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 0.781E-01 | 0.478E-01 |

Table 5.1.16 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

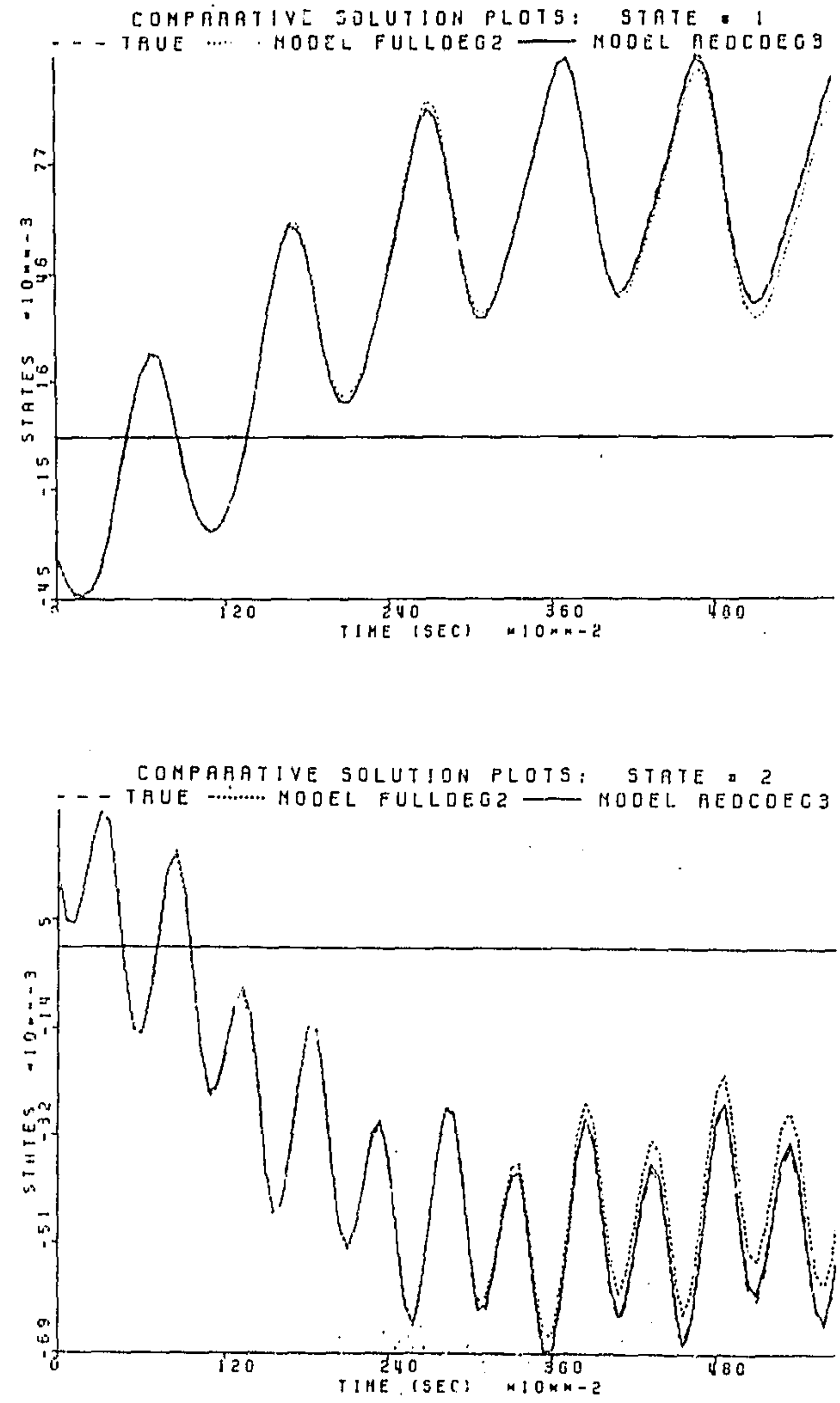



Figure 5.1.27 Simulation Number 26 of Table 5.1.16

```
************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 0.518E-01 | 0.282E-01 |
| 2 | 0.025 | 0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 0.566E-01 | 0.295E-01 |
| 3 | 0.025 | 0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 0.103 | 0.583E-01 |
| 4 | 0.025 | 0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 0.109 | 0.631E-01 |
| 5 | 0.025 | -0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 0.485E-01 | 0.303E-01 |
| 6 | 0.025 | -0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 0.497E-01 | 0.309E-01 |
| 7 | 0.025 | -0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 0.865E-01 | 0.578E-01 |
| 8 | 0.025 | -0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 0.922E-01 | 0.630E-01 |
| 9 | -0.025 | 0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 0.472E-01 | 0.280E-01 |
| 10 | -0.025 | 0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 0.501E-01 | 0.290E-01 |
| 11 | -0.025 | 0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 0.902E-01 | 0.563E-01 |
| 12 | -0.025 | 0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 0.957E-01 | 0.612E-01 |
| 13 | -0.025 | -0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 0.439E-01 | 0.298E-01 |
| 14 | -0.025 | -0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 0.442E-01 | 0.300E-01 |
| 15 | -0.025 | -0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 0.760E-01 | 0.553E-01 |
| 16 | -0.025 | -0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 0.822E-01 | 0.609E-01 |
| 17 | 0.025 | 0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.268E-01 | 0.158E-01 |
| 18 | 0.025 | 0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.333E-01 | 0.198E-01 |
| 19 | 0.025 | 0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.608E-01 | 0.450E-01 |
| 20 | 0.025 | 0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.676E-01 | 0.536E-01 |
| 21 | 0.025 | -0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.231E-01 | 0.147E-01 |
| 22 | 0.025 | -0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.271E-01 | 0.175E-01 |
| 23 | 0.025 | -0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.483E-01 | 0.399E-01 |
| 24 | 0.025 | -0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.572E-01 | 0.504E-01 |
| 25 | -0.025 | 0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.235E-01 | 0.143E-01 |
| 26 | -0.025 | 0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.285E-01 | 0.175E-01 |
| 27 | -0.025 | 0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.522E-01 | 0.411E-01 |
| 28 | -0.025 | 0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.600E-01 | 0.504E-01 |
| 29 | -0.025 | -0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.194E-01 | 0.122E-01 |
| 30 | -0.025 | -0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.223E-01 | 0.140E-01 |
| 31 | -0.025 | -0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.394E-01 | 0.336E-01 |
| 32 | -0.025 | -0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.493E-01 | 0.451E-01 |

Table 5.1.17 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

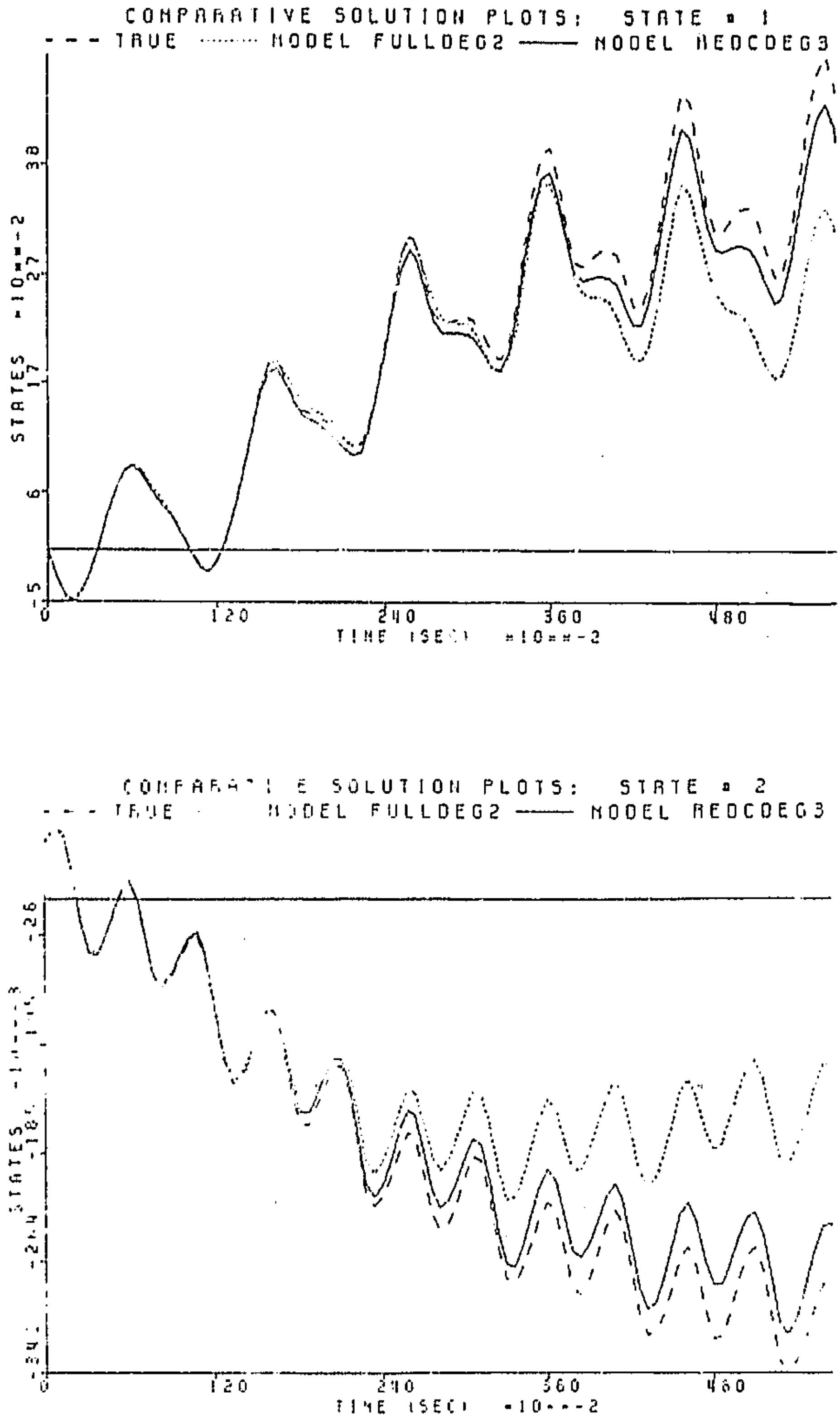



Figure 5.1.26 Simulation Number 4 of Table 5.1.17

```
***************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2: 13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
***************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.075 | 0.075 | 0.100 | 0.100 | 0.50 | 1.50 | 0.473E-02 | 0.661E-02 |
| 2 | 0.075 | 0.075 | 0.100 | -0.100 | 0.50 | 1.50 | 0.444E-02 | 0.617E-02 |
| 3 | 0.075 | 0.075 | -0.100 | 0.100 | 0.50 | 1.50 | 0.103 | 0.996E-01 |
| 4 | 0.075 | 0.075 | -0.100 | -0.100 | 0.50 | 1.50 | 0.994E-01 | 0.919E-01 |
| 5 | 0.075 | -0.075 | 0.100 | 0.100 | 0.50 | 1.50 | 0.665E-02 | 0.692E-02 |
| 6 | 0.075 | -0.075 | 0.100 | -0.100 | 0.50 | 1.50 | 0.635E-02 | 0.695E-02 |
| 7 | 0.075 | -0.075 | -0.100 | 0.100 | 0.50 | 1.50 | 0.172E-02 | 0.224E-02 |
| 8 | 0.075 | -0.075 | -0.100 | -0.100 | 0.50 | 1.50 | 0.165E-02 | 0.173E-02 |
| 9 | -0.075 | 0.075 | 0.100 | 0.100 | 0.50 | 1.50 | 0.231E-01 | 0.286E-01 |
| 10 | -0.075 | 0.075 | 0.100 | -0.100 | 0.50 | 1.50 | 0.242E-01 | 0.296E-01 |
| 11 | -0.075 | 0.075 | -0.100 | 0.100 | 0.50 | 1.50 | 0.854E-02 | 0.107E-01 |
| 12 | -0.075 | 0.075 | -0.100 | -0.100 | 0.50 | 1.50 | 0.465E-02 | 0.103E-01 |
| 13 | -0.075 | -0.075 | 0.100 | 0.100 | 0.50 | 1.50 | 0.428E-01 | 0.405E-01 |
| 14 | -0.075 | -0.075 | 0.100 | -0.100 | 0.50 | 1.50 | 0.429E-01 | 0.416E-01 |
| 15 | -0.075 | -0.075 | -0.100 | 0.100 | 0.50 | 1.50 | 0.302E-01 | 0.290E-01 |
| 16 | -0.075 | -0.075 | -0.100 | -0.100 | 0.50 | 1.50 | 0.290E-01 | 0.279E-01 |
| 17 | 0.075 | 0.075 | 0.200 | 0.200 | 0.50 | 1.50 | 0.655E-02 | 0.752E-02 |
| 18 | 0.075 | 0.075 | 0.200 | -0.200 | 0.50 | 1.50 | 0.787E-02 | 0.110E-01 |
| 19 | 0.075 | 0.075 | -0.200 | 0.200 | 0.50 | 1.50 | 0.344E-01 | 0.371E-01 |
| 20 | 0.075 | 0.075 | -0.200 | -0.200 | 0.50 | 1.50 | 0.231E-01 | 0.216E-01 |
| 21 | 0.075 | -0.075 | 0.200 | 0.200 | 0.50 | 1.50 | 0.168E-01 | 0.182E-01 |
| 22 | 0.075 | -0.075 | 0.200 | -0.200 | 0.50 | 1.50 | 0.150E-01 | 0.179E-01 |
| 23 | 0.075 | -0.075 | -0.200 | 0.200 | 0.50 | 1.50 | 0.130E-01 | 0.148E-01 |
| 24 | 0.075 | -0.075 | -0.200 | -0.200 | 0.50 | 1.50 | 0.154E-01 | 0.151E-01 |
| 25 | -0.075 | 0.075 | 0.200 | 0.200 | 0.50 | 1.50 | 0.263E-02 | 0.599E-02 |
| 26 | -0.075 | 0.075 | 0.200 | -0.200 | 0.50 | 1.50 | 0.168E-01 | 0.237E-01 |
| 27 | -0.075 | 0.075 | -0.200 | 0.200 | 0.50 | 1.50 | 0.836E-02 | 0.152E-01 |
| 28 | -0.075 | 0.075 | -0.200 | -0.200 | 0.50 | 1.50 | 0.722E-02 | 0.114E-01 |
| 29 | -0.075 | -0.075 | 0.200 | 0.200 | 0.50 | 1.50 | 0.575E-01 | 0.606E-01 |
| 30 | -0.075 | -0.075 | 0.200 | -0.200 | 0.50 | 1.50 | 0.566E-01 | 0.628E-01 |
| 31 | -0.075 | -0.075 | -0.200 | 0.200 | 0.50 | 1.50 | 0.751E-01 | 0.795E-01 |
| 32 | -0.075 | -0.075 | -0.200 | -0.200 | 0.50 | 1.50 | 0.781E-01 | 0.829E-01 |
| 33 | 0.075 | 0.075 | 0.300 | 0.300 | 0.50 | 1.50 | 0.201E-01 | 0.343E-01 |
| 34 | 0.075 | 0.075 | 0.300 | -0.300 | 0.50 | 1.50 | 0.528E-01 | 0.469E-01 |
| 35 | 0.075 | 0.075 | -0.300 | 0.300 | 0.50 | 1.50 | 0.385E-01 | 0.479E-01 |
| 36 | 0.075 | 0.075 | -0.300 | -0.300 | 0.50 | 1.50 | 0.267E-01 | 0.351E-01 |
| 37 | 0.075 | -0.075 | 0.300 | 0.300 | 0.50 | 1.50 | 0.498E-01 | 0.573E-01 |
| 38 | 0.075 | -0.075 | 0.300 | -0.300 | 0.50 | 1.50 | 0.379E-01 | 0.445E-01 |
| 39 | 0.075 | -0.075 | -0.300 | 0.300 | 0.50 | 1.50 | 0.697E-01 | 0.751E-01 |
| 40 | 0.075 | -0.075 | -0.300 | -0.300 | 0.50 | 1.50 | 0.830E-01 | 0.895E-01 |
| 41 | -0.075 | 0.075 | 0.300 | 0.300 | 0.50 | 1.50 | 0.272E-01 | 0.362E-01 |
| 42 | -0.075 | 0.075 | 0.300 | -0.300 | 0.50 | 1.50 | 0.298E-01 | 0.330E-01 |
| 43 | -0.075 | 0.075 | -0.300 | 0.300 | 0.50 | 1.50 | 0.197E-01 | 0.356E-01 |
| 44 | -0.075 | 0.075 | -0.300 | -0.300 | 0.50 | 1.50 | 0.254E-01 | 0.426E-01 |
| 45 | -0.075 | -0.075 | 0.300 | 0.300 | 0.50 | 1.50 | 0.113 | 0.128 |
| 46 | -0.075 | -0.075 | 0.300 | -0.300 | 0.50 | 1.50 | 0.969E-01 | 0.115 |
| 47 | -0.075 | -0.075 | -0.300 | 0.300 | 0.50 | 1.50 | 0.168 | 0.204 |
| 48 | -0.075 | -0.075 | -0.300 | -0.300 | 0.50 | 1.50 | 0.180 | 0.223 |

Table 5.1.18 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

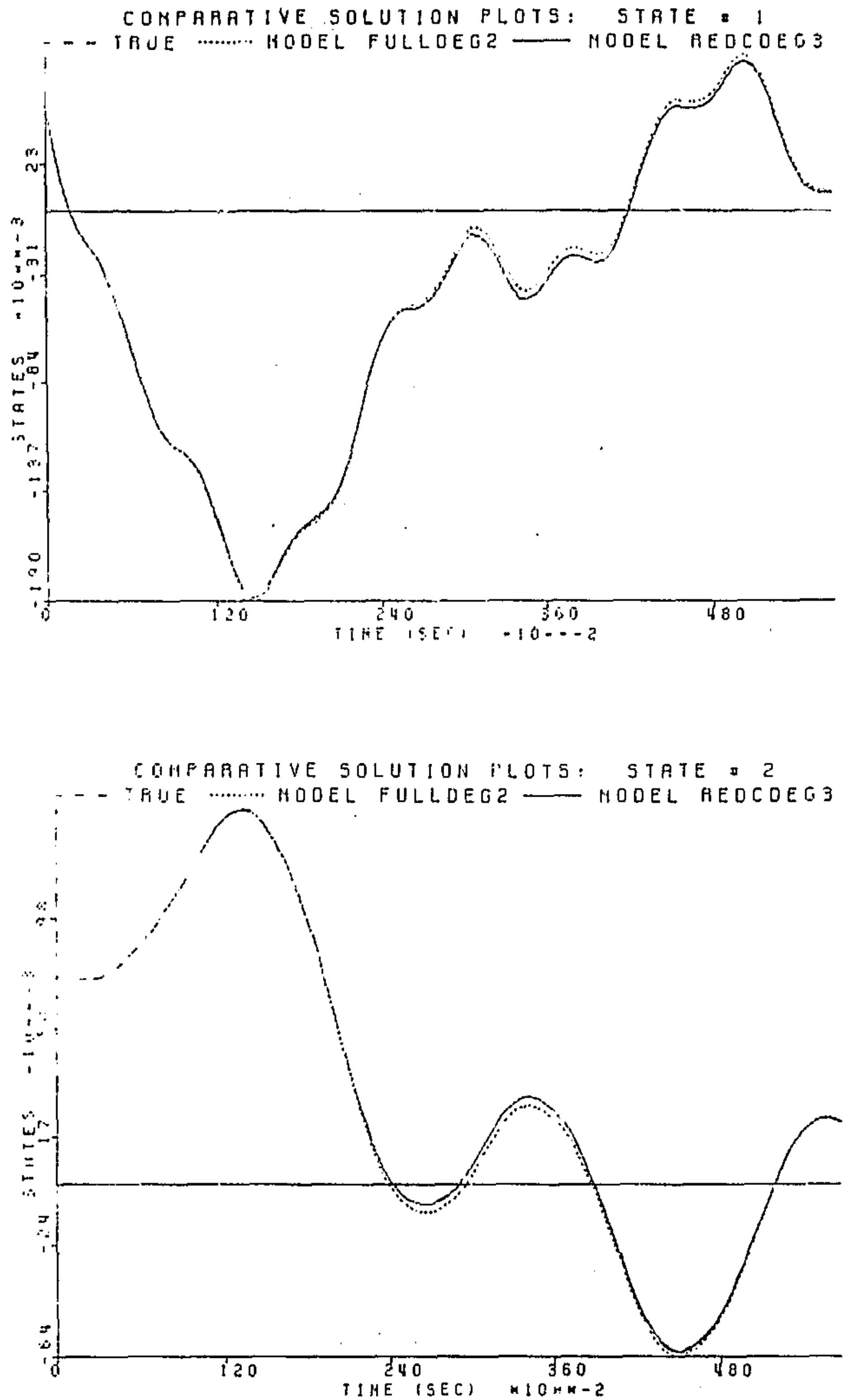
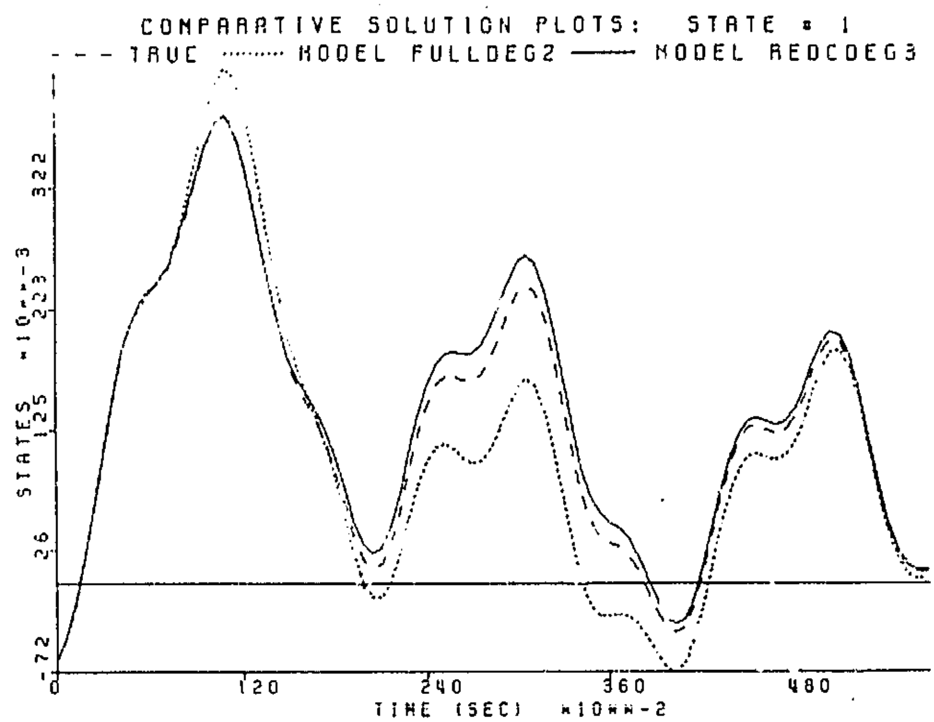



Figure 5.1.27 Simulation Number 2 of Table 5.1.18



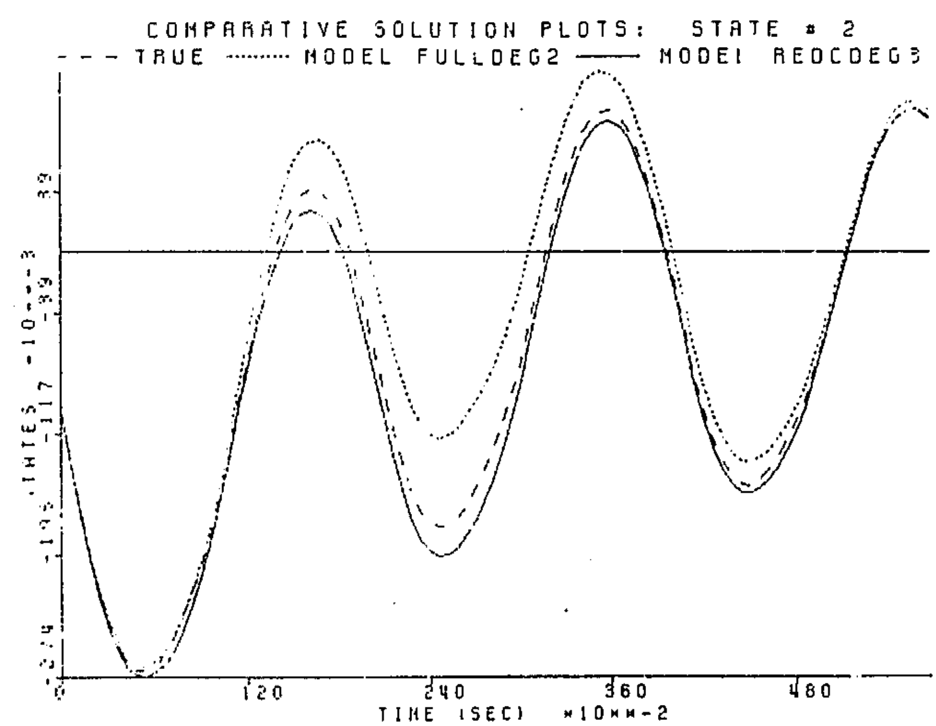




Figure 5.1.28 Simulation Number 46 of Table 5.1.18

```
**************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.050 | -0.050 | -0.107 | 0.251 | 3.43 | 3.75 | 0.127 | 0.111 |
| 2 | 0.050 | -0.050 | -0.015 | -0.088 | 2.63 | 2.13 | 0.661E-02 | 0.642E-02 |
| 3 | 0.050 | -0.050 | -0.020 | 0.172 | 4.05 | 4.91 | 0.705E-02 | 0.623E-02 |
| 4 | 0.050 | -0.050 | 0.217 | 0.223 | 4.82 | 2.08 | 0.558E-01 | 0.309E-01 |
| 5 | 0.050 | -0.050 | 0.364 | 0.141 | 4.84 | 3.37 | 0.863E-01 | 0.576E-01 |
| 6 | 0.050 | -0.050 | -0.304 | -0.014 | 4.12 | 4.32 | 0.794E-01 | 0.454E-01 |
| 7 | 0.050 | -0.050 | 0.127 | 0.265 | 3.32 | 5.34 | 0.177E-01 | 0.125E-01 |
| 8 | 0.050 | -0.050 | 0.152 | -0.298 | 3.54 | 4.50 | 0.224E-01 | 0.167E-01 |
| 9 | 0.050 | -0.050 | -0.256 | 0.238 | 2.49 | 2.46 | 0.289 | 0.181 |
| 10 | 0.050 | -0.050 | -0.335 | -0.342 | 4.35 | 4.70 | 0.958E-01 | 0.606E-01 |
| 11 | 0.050 | -0.050 | -0.032 | -0.094 | 4.14 | 4.81 | 0.271E-02 | 0.433E-02 |
| 12 | 0.050 | -0.050 | 0.072 | 0.193 | 2.90 | 2.83 | 1.79 | 2.44 |
| 13 | 0.050 | -0.050 | -0.281 | 0.078 | 4.69 | 2.89 | 0.727E-01 | 0.410E-01 |
| 14 | 0.050 | -0.050 | -0.196 | 0.292 | 4.07 | 2.98 | 0.442E-01 | 0.267E-01 |
| 15 | 0.050 | -0.050 | 0.382 | -0.203 | 4.92 | 3.21 | 0.865E-01 | 0.616E-01 |
| 16 | 0.050 | -0.050 | 0.060 | 0.090 | 4.93 | 5.41 | 0.128E-01 | 0.150E-01 |
| 17 | 0.050 | -0.050 | 0.022 | -0.083 | 3.92 | 5.43 | 0.627E-02 | 0.462E-02 |
| 18 | 0.050 | -0.050 | 0.396 | -0.391 | 4.78 | 5.05 | 0.108 | 0.748E-01 |
| 19 | 0.050 | -0.050 | -0.191 | -0.309 | 4.58 | 5.53 | 0.405E-01 | 0.251E-01 |
| 20 | 0.050 | -0.050 | -0.316 | -0.241 | 2.86 | 2.39 | 0.737E-01 | 0.454E-01 |
| 21 | 0.050 | -0.050 | -0.317 | -0.052 | 3.77 | 2.47 | 0.781E-01 | 0.462E-01 |
| 22 | 0.050 | -0.050 | 0.161 | -0.124 | 4.16 | 4.24 | 0.178 | 0.117 |
| 23 | 0.050 | -0.050 | 0.140 | -0.246 | 5.05 | 2.81 | 0.225E-01 | 0.152E-01 |
| 24 | 0.050 | -0.050 | -0.125 | -0.047 | 5.44 | 3.97 | 0.247E-01 | 0.169E-01 |
| 25 | 0.050 | -0.050 | 0.335 | 0.397 | 3.21 | 4.90 | 0.644E-01 | 0.395E-01 |
| 26 | 0.050 | -0.050 | -0.054 | -0.218 | 5.31 | 3.38 | 0.238E-02 | 0.361E-02 |
| 27 | 0.050 | -0.050 | -0.319 | 0.077 | 5.57 | 4.00 | 0.805E-01 | 0.482E-01 |
| 28 | 0.050 | -0.050 | -0.139 | -0.229 | 2.30 | 3.57 | 0.231E-01 | 0.161E-01 |
| 29 | 0.050 | -0.050 | 0.280 | 0.128 | 2.23 | 2.41 | 0.116 | 0.637E-01 |
| 30 | 0.050 | -0.050 | 0.055 | 0.124 | 5.93 | 3.36 | 0.179E-02 | 0.228E-02 |
| 31 | 0.050 | -0.050 | 0.141 | -0.087 | 2.91 | 3.16 | 0.382E-01 | 0.284E-01 |
| 32 | 0.050 | -0.050 | -0.037 | 0.369 | 2.31 | 2.37 | 0.325 | 0.232 |
| 33 | 0.050 | -0.050 | 0.143 | -0.014 | 4.99 | 3.11 | 0.317E-01 | 0.205E-01 |
| 34 | 0.050 | -0.050 | -0.350 | 0.139 | 5.40 | 3.31 | 0.846E-01 | 0.545E-01 |
| 35 | 0.050 | -0.050 | 0.084 | 0.307 | 4.11 | 4.78 | 0.167E-01 | 0.238E-01 |
| 36 | 0.050 | -0.050 | -0.361 | -0.371 | 3.00 | 4.89 | 0.680E-01 | 0.442E-01 |
| 37 | 0.050 | -0.050 | -0.201 | 0.400 | 4.80 | 2.85 | 0.416E-01 | 0.241E-01 |
| 38 | 0.050 | -0.050 | -0.133 | -0.185 | 5.11 | 3.50 | 0.259E-01 | 0.176E-01 |
| 39 | 0.050 | -0.050 | 0.017 | 0.260 | 2.38 | 3.32 | 0.378E-02 | 0.430E-02 |
| 40 | 0.050 | -0.050 | 0.123 | 0.256 | 4.66 | 2.12 | 0.260E-01 | 0.188E-01 |
| 41 | 0.050 | -0.050 | 0.020 | -0.138 | 4.99 | 5.66 | 0.101E-01 | 0.875E-02 |
| 42 | 0.050 | -0.050 | 0.100 | -0.381 | 4.90 | 3.79 | 0.456E-02 | 0.788E-02 |
| 43 | 0.050 | -0.050 | -0.292 | 0.080 | 5.01 | 3.90 | 0.750E-01 | 0.431E-01 |
| 44 | 0.050 | -0.050 | 0.129 | -0.155 | 2.80 | 2.16 | 0.190E-01 | 0.153E-01 |
| 45 | 0.050 | -0.050 | 0.130 | -0.330 | 3.10 | 5.11 | 0.143E-01 | 0.103E-01 |
| 46 | 0.050 | -0.050 | -0.350 | 0.296 | 2.07 | 3.35 | 0.626E-01 | 0.376E-01 |
| 47 | 0.050 | -0.050 | 0.201 | -0.091 | 5.22 | 3.65 | 0.527E-01 | 0.300E-01 |
| 48 | 0.050 | -0.050 | -0.027 | -0.011 | 4.76 | 2.88 | 0.458E-02 | 0.326E-02 |
| 49 | 0.050 | -0.050 | 0.265 | 0.019 | 2.02 | 3.20 | 0.618E-01 | 0.333E-01 |
| 50 | 0.050 | -0.050 | -0.119 | 0.171 | 4.24 | 2.59 | 0.197E-01 | 0.140E-01 |

Table 5.1.19 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

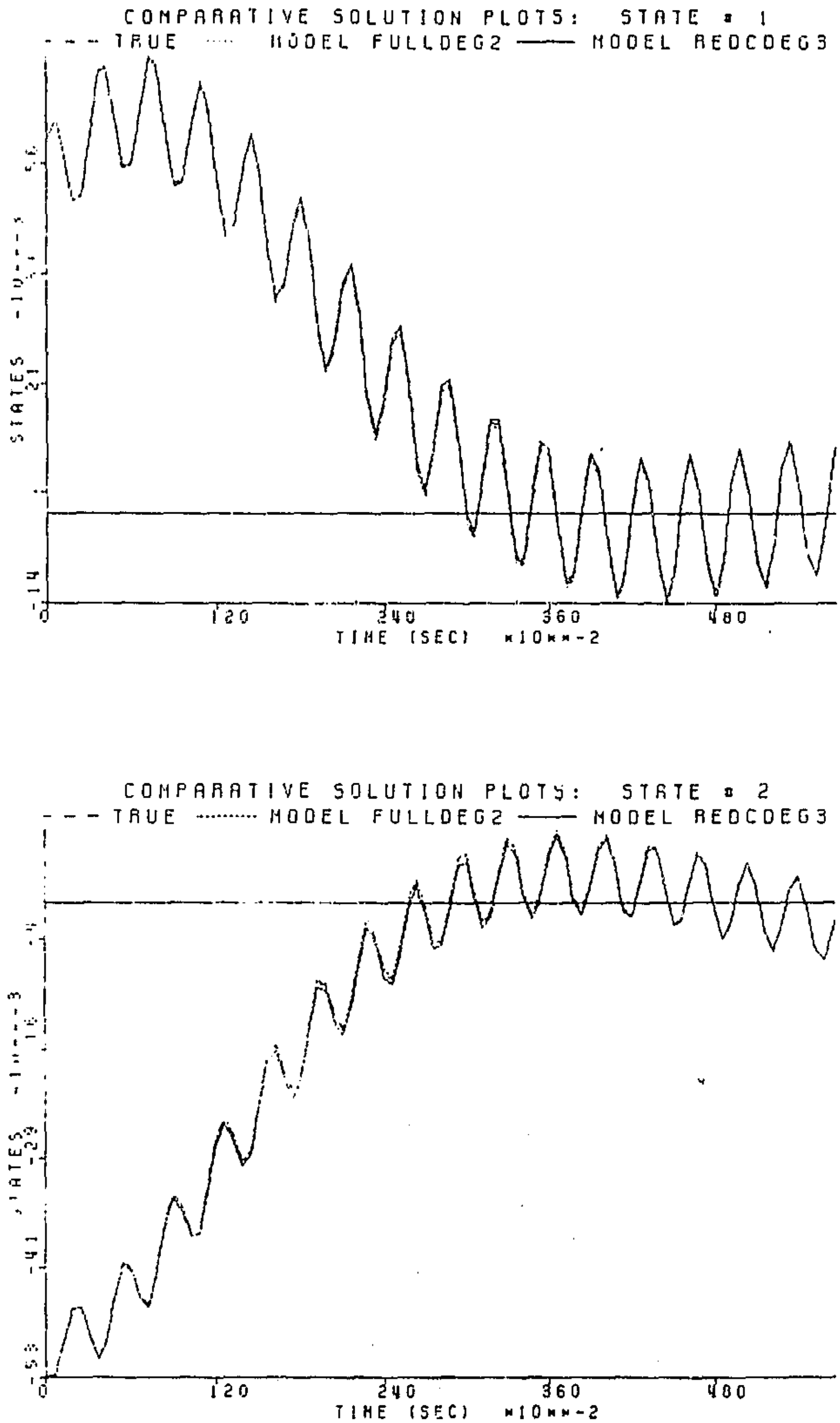
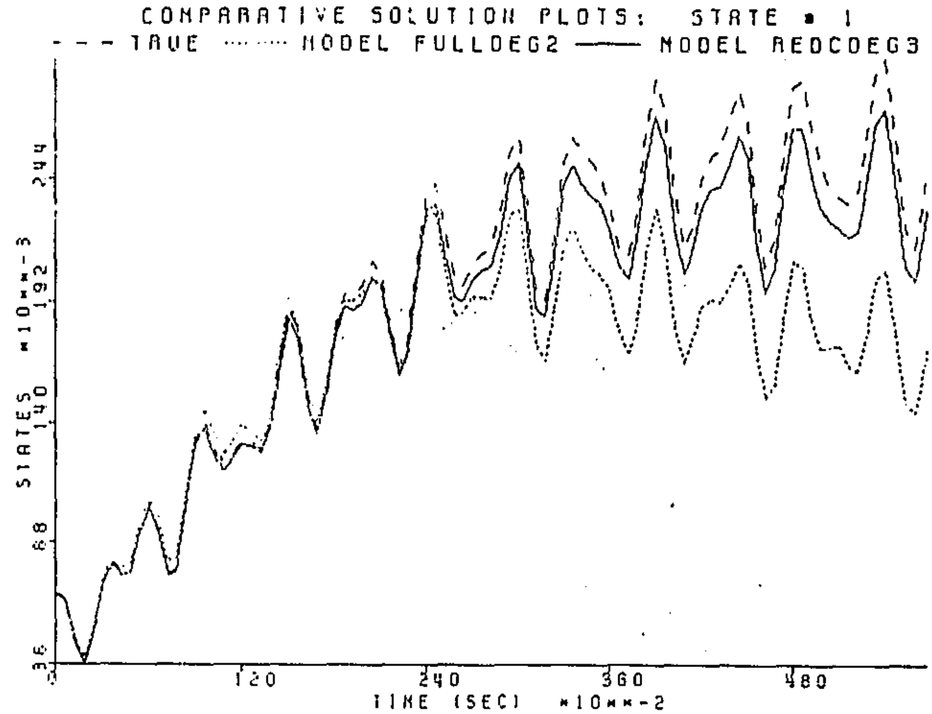



Figure 5.1.29 Simulation Number 12 of Table 5.1.19



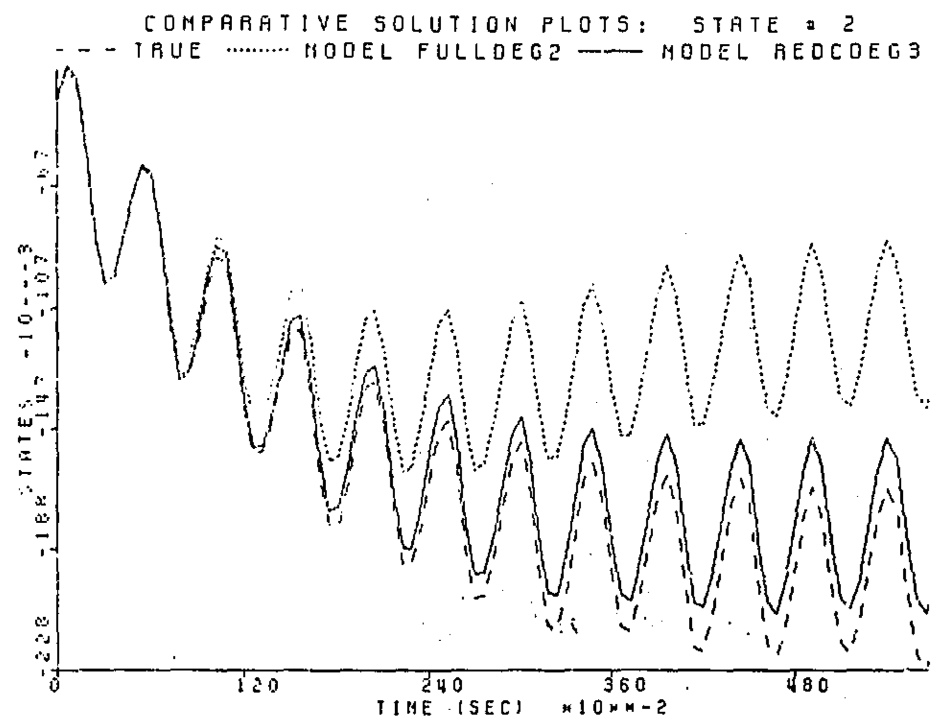
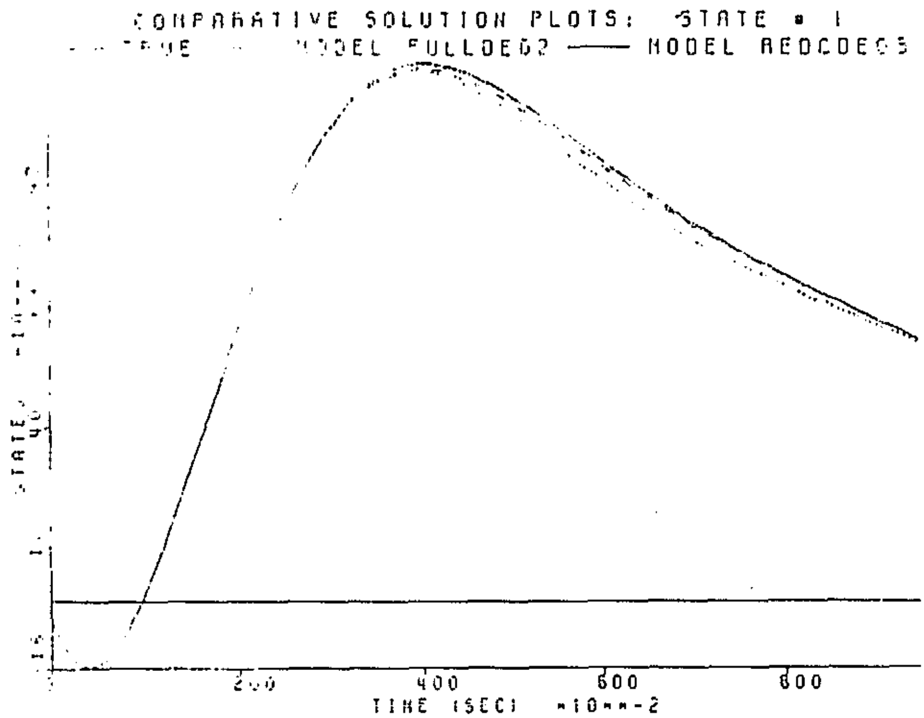




Figure 5.1.30 Simulation Number 46 of Table 5.1.19

```
***************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
***************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | -0.001 | 0.010 | -0.075 | 0.00 | 0.00 | 0.304 | 0.161 |
| 2 | 0.001 | -0.001 | 0.010 | -0.075 | 0.01 | 0.01 | 0.259 | 0.140 |
| 3 | 0.001 | -0.001 | 0.010 | -0.075 | 0.02 | 0.02 | 0.180 | 0.111 |
| 4 | 0.001 | -0.001 | 0.010 | -0.075 | 0.05 | 0.05 | 0.993E-01 | 0.767E-01 |
| 5 | 0.001 | -0.001 | -0.050 | 0.050 | 0.00 | 0.00 | 0.748E-01 | 0.220E-01 |
| 6 | 0.001 | -0.001 | -0.050 | 0.050 | 0.01 | 0.01 | 0.658E-01 | 0.212E-01 |
| 7 | 0.001 | -0.001 | -0.050 | 0.050 | 0.02 | 0.02 | 0.348E-01 | 0.121E-01 |
| 8 | 0.001 | -0.001 | -0.050 | 0.050 | 0.05 | 0.05 | 0.587E-01 | 0.179E-01 |
| 9 | 0.001 | -0.001 | -0.075 | 0.010 | 0.00 | 0.00 | 0.974E-01 | 0.226E-01 |
| 10 | 0.001 | -0.001 | -0.075 | 0.010 | 0.01 | 0.01 | 0.880E-01 | 0.230E-01 |
| 11 | 0.001 | -0.001 | -0.075 | 0.010 | 0.02 | 0.02 | 0.659E-01 | 0.203E-01 |
| 12 | 0.001 | -0.001 | -0.075 | 0.010 | 0.05 | 0.05 | 0.833E-01 | 0.145E-01 |
| 13 | 0.001 | -0.001 | -0.075 | -0.075 | 0.00 | 0.00 | 0.118 | 0.527E-01 |
| 14 | 0.001 | -0.001 | -0.075 | -0.075 | 0.01 | 0.01 | 0.970E-01 | 0.469E-01 |
| 15 | 0.001 | -0.001 | -0.075 | -0.075 | 0.02 | 0.02 | 0.586E-01 | 0.367E-01 |
| 16 | 0.001 | -0.001 | -0.075 | -0.075 | 0.05 | 0.05 | 0.345E-01 | 0.114E-01 |
| 17 | 0.010 | -0.010 | 0.010 | -0.075 | 0.00 | 0.00 | 0.305 | 0.160 |
| 18 | 0.010 | -0.010 | 0.010 | -0.075 | 0.01 | 0.01 | 0.260 | 0.139 |
| 19 | 0.010 | -0.010 | 0.010 | -0.075 | 0.02 | 0.02 | 0.180 | 0.110 |
| 20 | 0.010 | -0.010 | 0.010 | -0.075 | 0.05 | 0.05 | 0.101 | 0.760E-01 |
| 21 | 0.010 | -0.010 | -0.050 | 0.050 | 0.00 | 0.00 | 0.689E-01 | 0.207E-01 |
| 22 | 0.010 | -0.010 | -0.050 | 0.050 | 0.01 | 0.01 | 0.590E-01 | 0.198E-01 |
| 23 | 0.010 | -0.010 | -0.050 | 0.050 | 0.02 | 0.02 | 0.294E-01 | 0.105E-01 |
| 24 | 0.010 | -0.010 | -0.050 | 0.050 | 0.05 | 0.05 | 0.502E-01 | 0.169E-01 |
| 25 | 0.010 | -0.010 | -0.075 | 0.010 | 0.00 | 0.00 | 0.912E-01 | 0.215E-01 |
| 26 | 0.010 | -0.010 | -0.075 | 0.010 | 0.01 | 0.01 | 0.820E-01 | 0.217E-01 |
| 27 | 0.010 | -0.010 | -0.075 | 0.010 | 0.02 | 0.02 | 0.604E-01 | 0.186E-01 |
| 28 | 0.010 | -0.010 | -0.075 | 0.010 | 0.05 | 0.05 | 0.749E-01 | 0.134E-01 |
| 29 | 0.010 | -0.010 | -0.075 | -0.075 | 0.00 | 0.00 | 0.118 | 0.528E-01 |
| 30 | 0.010 | -0.010 | -0.075 | -0.075 | 0.01 | 0.01 | 0.966E-01 | 0.468E-01 |
| 31 | 0.010 | -0.010 | -0.075 | -0.075 | 0.02 | 0.02 | 0.564E-01 | 0.355E-01 |
| 32 | 0.010 | -0.010 | -0.075 | -0.075 | 0.05 | 0.05 | 0.341E-01 | 0.102E-01 |

Table 5.1.20 Low Frequency Table for Second Degree Full Model versus Third Degree Reduced Model



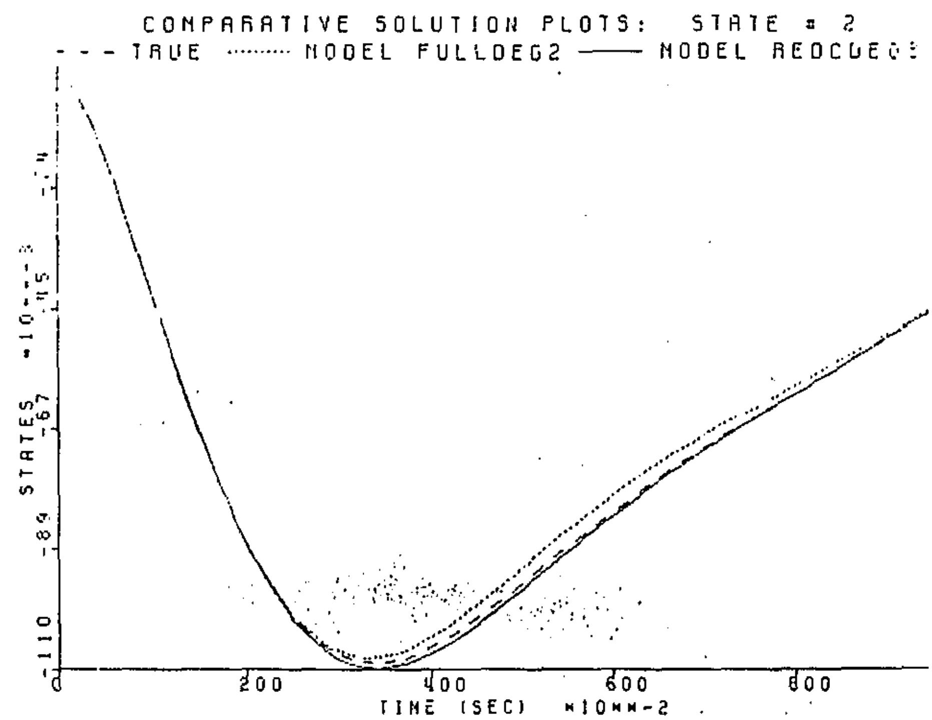
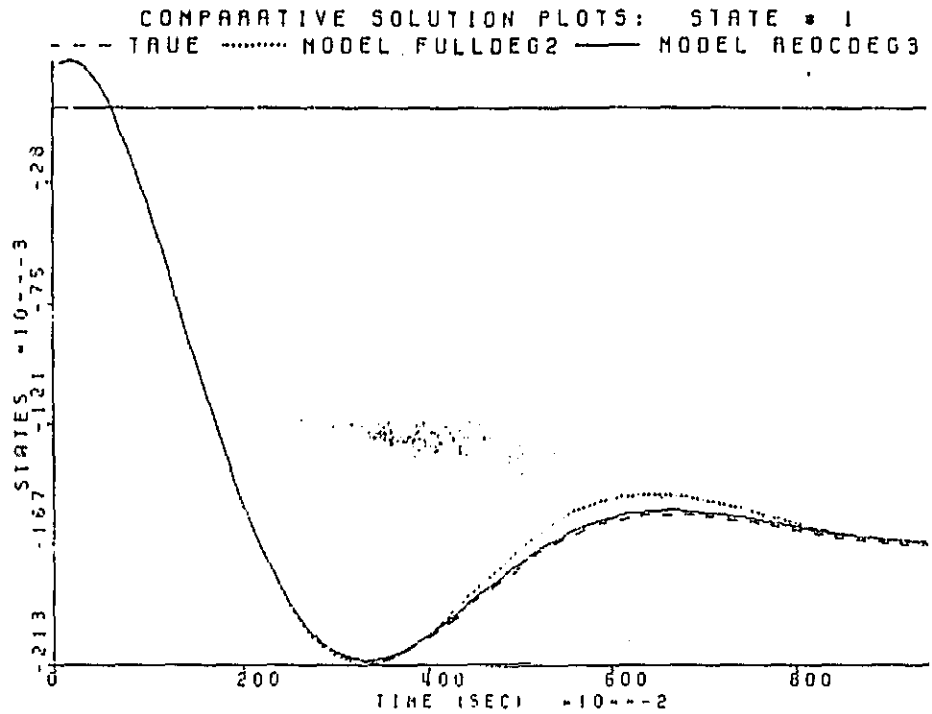




Figure 5.1.31 Simulation Number 3 of Table 5.1.20



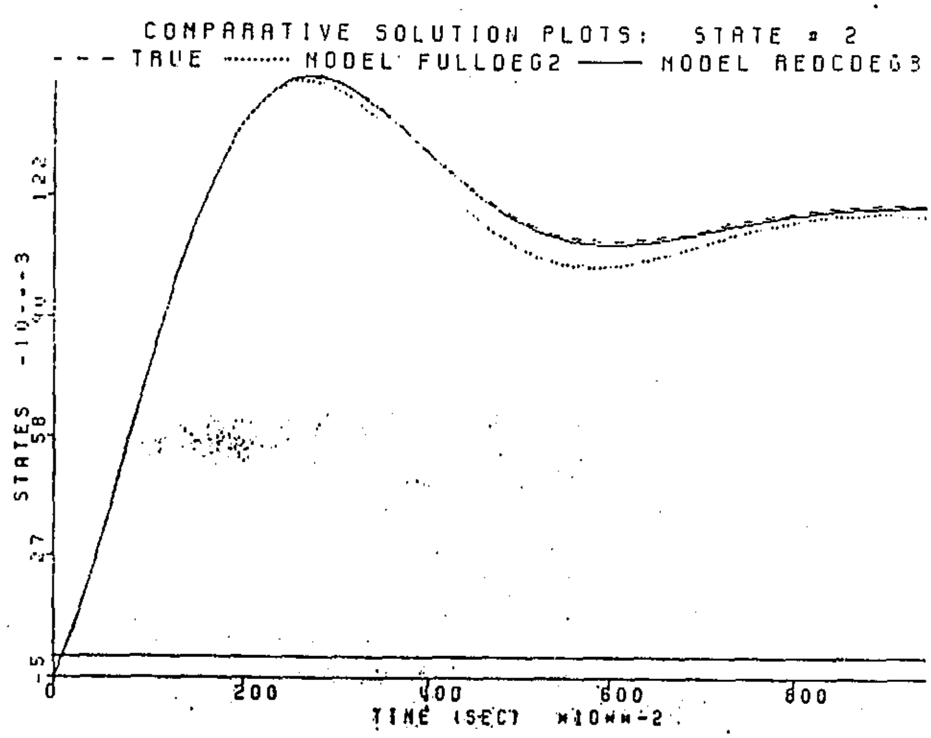
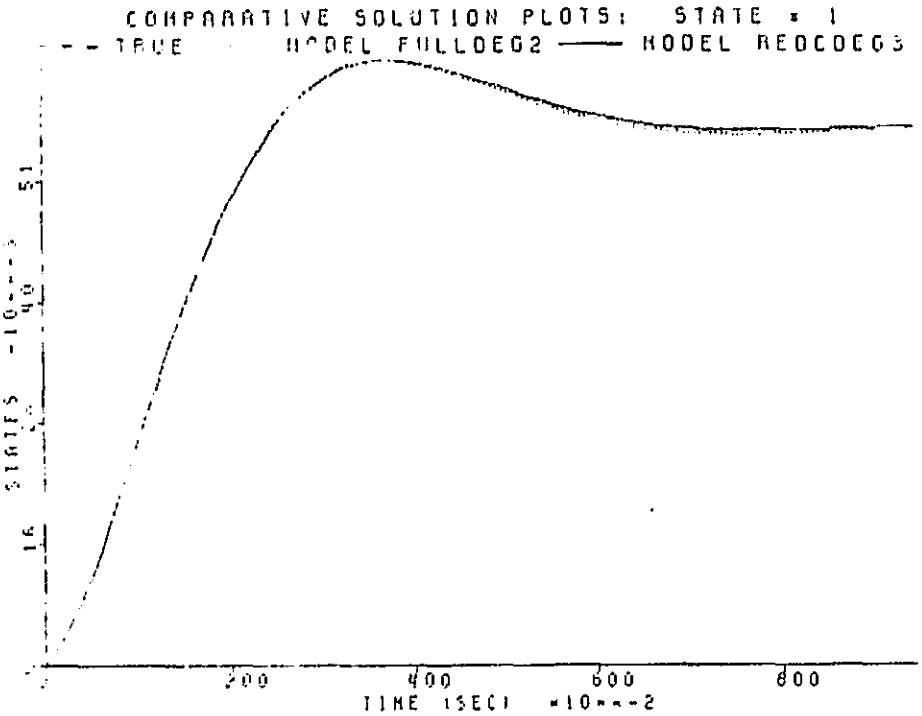




Figure 5.1.32 Simulation Number 21 of Table 5.1.20

*************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2: 13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.000 | 0.010 | -0.022 | -0.008 | 0.00 | 0.00 | 4.29 | 7.10 |
| 2 | -0.006 | 0.004 | 0.008 | -0.011 | 0.00 | 0.00 | 0.721 | 0.424 |
| 3 | 0.009 | 0.006 | -0.010 | 0.017 | 0.00 | 0.00 | 1.15 | 0.737 |
| 4 | 0.006 | 0.010 | 0.022 | -0.010 | 0.00 | 0.00 | 0.756E-02 | 0.405E-02 |
| 5 | 0.006 | -0.007 | -0.018 | -0.003 | 0.00 | 0.00 | 3.59 | 4.02 |
| 6 | -0.004 | 0.001 | -0.021 | -0.005 | 0.00 | 0.00 | 6.46 | 10.7 |
| 7 | 0.006 | 0.003 | 0.012 | 0.018 | 0.00 | 0.00 | 1.96 | 1.79 |
| 8 | -0.009 | -0.003 | -0.021 | 0.024 | 0.00 | 0.00 | 4.47 | 4.25 |
| 9 | -0.000 | 0.003 | 0.004 | 0.003 | 0.00 | 0.00 | 0.340E-01 | 0.194E-01 |
| 10 | -0.000 | 0.002 | -0.009 | -0.013 | 0.00 | 0.00 | 0.432 | 0.334 |
| 11 | 0.009 | 0.001 | 0.011 | 0.011 | 0.00 | 0.00 | 0.792 | 0.796 |
| 12 | 0.002 | -0.008 | 0.017 | 0.002 | 0.00 | 0.00 | 0.117E-01 | 0.466E-02 |
| 13 | 0.005 | -0.008 | -0.019 | -0.010 | 0.00 | 0.00 | 2.74 | 3.75 |
| 14 | -0.004 | -0.005 | 0.022 | -0.018 | 0.00 | 0.00 | 0.237E-02 | 0.146E-02 |
| 15 | -0.007 | 0.007 | -0.001 | -0.020 | 0.00 | 0.00 | 35.1 | 41.4 |
| 16 | -0.010 | 0.005 | -0.005 | -0.006 | 0.00 | 0.00 | 0.275 | 0.189 |
| 17 | -0.007 | 0.006 | -0.018 | 0.009 | 0.00 | 0.00 | 5.37 | 5.41 |
| 18 | -0.001 | -0.004 | 0.015 | 0.021 | 0.00 | 0.00 | 4.07 | 5.92 |
| 19 | 0.008 | -0.005 | 0.007 | 0.017 | 0.00 | 0.00 | 4.12 | 3.51 |
| 20 | 0.003 | -0.003 | -0.016 | -0.008 | 0.00 | 0.00 | 1.32 | 1.15 |
| 21 | 0.004 | -0.003 | 0.019 | 0.006 | 0.00 | 0.00 | 0.391E-02 | 0.151E-02 |
| 22 | -0.003 | 0.006 | -0.010 | 0.007 | 0.00 | 0.00 | 1.25 | 0.789 |
| 23 | 0.007 | -0.002 | 0.014 | -0.012 | 0.00 | 0.00 | 0.575E-01 | 0.309E-01 |
| 24 | -0.003 | -0.004 | 0.010 | -0.009 | 0.00 | 0.00 | 0.258 | 0.142 |
| 25 | -0.000 | -0.009 | -0.001 | -0.013 | 0.00 | 0.00 | 46.4 | 21.0 |
| 26 | -0.009 | 0.007 | 0.007 | -0.015 | 0.00 | 0.00 | 0.887 | 0.556 |
| 27 | -0.002 | 0.002 | -0.006 | -0.023 | 0.00 | 0.00 | 40.2 | 34.2 |
| 28 | 0.009 | 0.002 | -0.012 | 0.012 | 0.00 | 0.00 | 1.59 | 1.07 |
| 29 | -0.006 | -0.007 | -0.003 | 0.006 | 0.00 | 0.00 | 0.956 | 0.616 |
| 30 | 0.004 | -0.000 | -0.000 | -0.009 | 0.00 | 0.00 | 18.9 | 11.0 |
| 31 | 0.001 | -0.007 | 0.022 | 0.014 | 0.00 | 0.00 | 0.395E-01 | 0.198E-01 |
| 32 | 0.008 | -0.002 | 0.004 | 0.024 | 0.00 | 0.00 | 2.89 | 1.93 |
| 33 | 0.001 | -0.007 | -0.023 | 0.018 | 0.00 | 0.00 | 4.64 | 3.78 |
| 34 | 0.003 | -0.005 | -0.024 | 0.024 | 0.00 | 0.00 | 2.39 | 1.48 |
| 35 | -0.001 | -0.007 | 0.017 | 0.012 | 0.00 | 0.00 | 0.242 | 0.119 |
| 36 | -0.010 | -0.006 | 0.015 | 0.014 | 0.00 | 0.00 | 1.88 | 1.49 |
| 37 | -0.009 | -0.006 | -0.004 | 0.002 | 0.00 | 0.00 | 0.637 | 0.415 |
| 38 | 0.001 | 0.008 | -0.005 | 0.020 | 0.00 | 0.00 | 0.743 | 0.449 |
| 39 | 0.000 | -0.001 | -0.004 | 0.025 | 0.00 | 0.00 | 0.592 | 0.376 |
| 40 | 0.003 | -0.009 | -0.011 | 0.009 | 0.00 | 0.00 | 1.26 | 0.832 |
| 41 | -0.000 | 0.004 | 0.011 | -0.006 | 0.00 | 0.00 | 0.362 | 0.192 |
| 42 | 0.008 | -0.005 | -0.010 | 0.010 | 0.00 | 0.00 | 1.16 | 0.742 |
| 43 | -0.009 | -0.008 | 0.012 | -0.021 | 0.00 | 0.00 | 0.236E-01 | 0.148E-01 |
| 44 | 0.003 | -0.007 | -0.021 | 0.015 | 0.00 | 0.00 | 6.64 | 8.19 |
| 45 | 0.001 | 0.007 | -0.016 | 0.012 | 0.00 | 0.00 | 3.74 | 3.24 |
| 46 | -0.008 | 0.004 | 0.005 | 0.015 | 0.00 | 0.00 | 13.7 | 14.6 |
| 47 | -0.001 | -0.002 | 0.015 | 0.010 | 0.00 | 0.00 | 0.892 | 0.469 |
| 48 | -0.007 | 0.009 | 0.018 | -0.002 | 0.00 | 0.00 | 0.176E-01 | 0.819E-02 |
| 49 | 0.006 | -0.005 | 0.004 | 0.008 | 0.00 | 0.00 | 2.03 | 1.60 |
| 50 | -0.006 | 0.003 | -0.011 | -0.021 | 0.00 | 0.00 | 4.09 | 4.17 |

Tabie 5.1.21 Step Response Table for Second Degree Full Model versus Third Degree Reduced Model



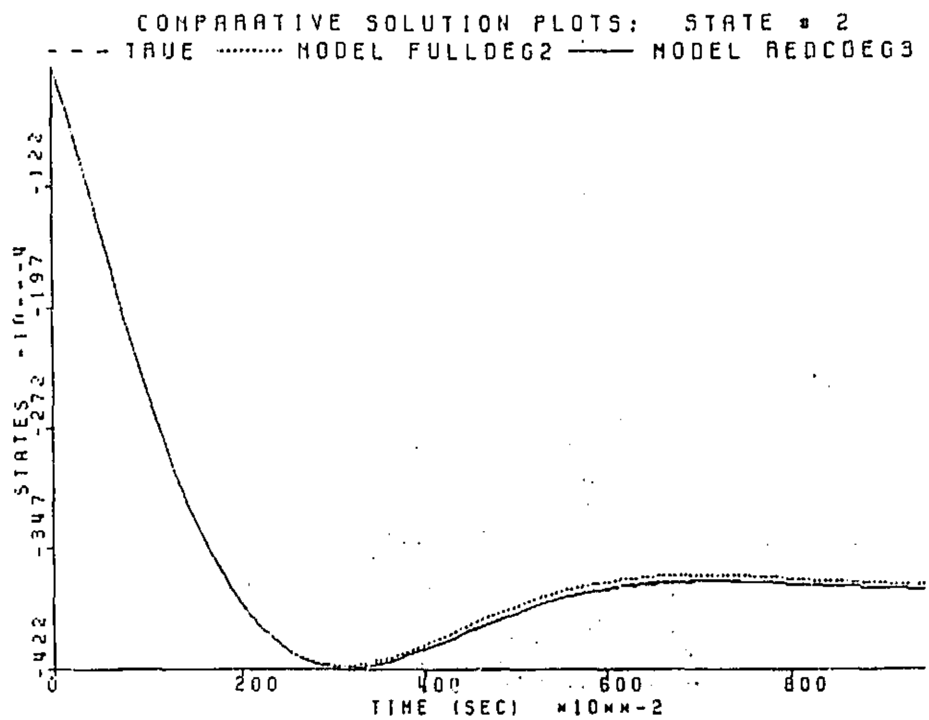




Figure 5.1.33 Simulation Number 15 of Table 5.1.21

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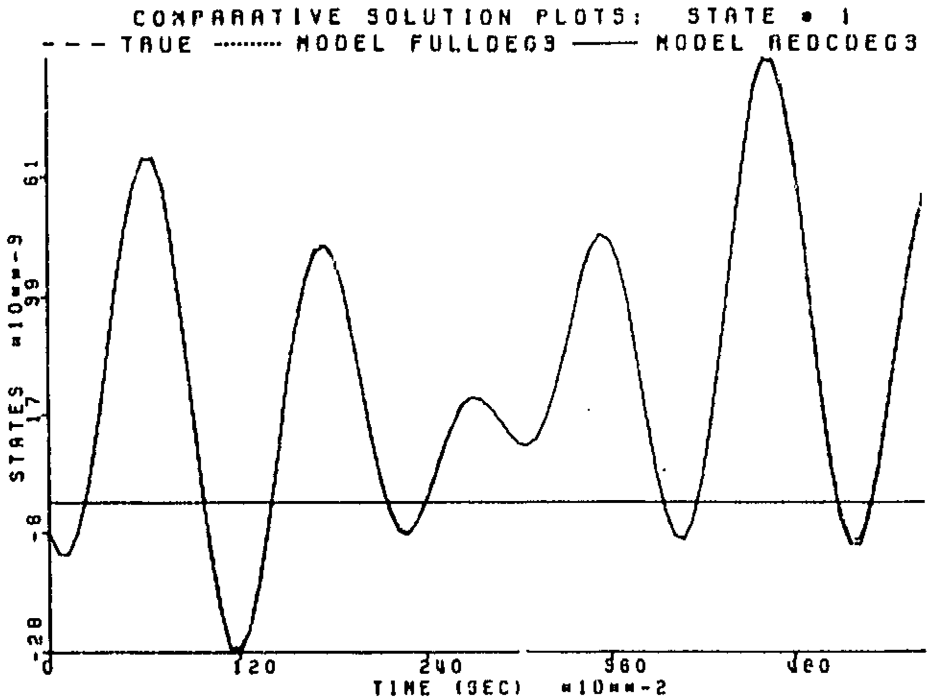
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.240E-01 | 0.241E-01 |
| 2 | 0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.487 | 1.85 |
| 3 | 0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.859 | 2.27 |
| 4 | 0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.497 | 2.30 |
| 5 | 0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.711 | 2.00 |
| 6 | 0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 11.1 | 27.6 |
| 7 | 0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 18.1 | 64.9 |
| 8 | 0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 16.8 | 41.7 |
| 9 | 0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 8.19 | 16.0 |
| 10 | 0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.239E-01 | 0.195E-01 |
| 11 | 0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.758 | 4.16 |
| 12 | 0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 1.08 | 3.70 |
| 13 | 0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.805 | 5.26 |
| 14 | 0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.782 | 3.35 |
| 15 | 0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 13.4 | 32.5 |
| 16 | 0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 17.5 | 61.3 |
| 17 | 0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 20.1 | 48.9 |
| 18 | 0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 8.44 | 17.4 |
| 19 | -0.001 | -0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.378E-01 | 0.382E-01 |
| 20 | -0.001 | -0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.870 | 5.98 |
| 21 | -0.001 | -0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 1.14 | 4.57 |
| 22 | -0.001 | -0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.903 | 6.05 |
| 23 | -0.001 | -0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.726 | 3.25 |
| 24 | -0.001 | -0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 14.6 | 34.9 |
| 25 | -0.001 | -0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 16.9 | 57.9 |
| 26 | -0.001 | -0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 21.0 | 50.9 |
| 27 | -0.001 | -0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 8.19 | 16.8 |
| 28 | -0.001 | 0.001 | 0.000 | 0.000 | 0.75 | 1.00 | 0.183E-01 | 0.149E-01 |
| 29 | -0.001 | 0.001 | 0.050 | 0.050 | 0.75 | 1.00 | 0.556 | 2.50 |
| 30 | -0.001 | 0.001 | 0.050 | -0.050 | 0.75 | 1.00 | 0.938 | 2.89 |
| 31 | -0.001 | 0.001 | -0.050 | -0.050 | 0.75 | 1.00 | 0.573 | 2.74 |
| 32 | -0.001 | 0.001 | -0.050 | 0.050 | 0.75 | 1.00 | 0.671 | 2.05 |
| 33 | -0.001 | 0.001 | 0.150 | 0.150 | 0.75 | 1.00 | 12.2 | 30.0 |
| 34 | -0.001 | 0.001 | 0.150 | -0.150 | 0.75 | 1.00 | 17.6 | 61.3 |
| 35 | -0.001 | 0.001 | -0.150 | -0.150 | 0.75 | 1.00 | 17.6 | 43.6 |
| 36 | -0.001 | 0.001 | -0.150 | 0.150 | 0.75 | 1.00 | 7.94 | 15.5 |

Table 5.1.22 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



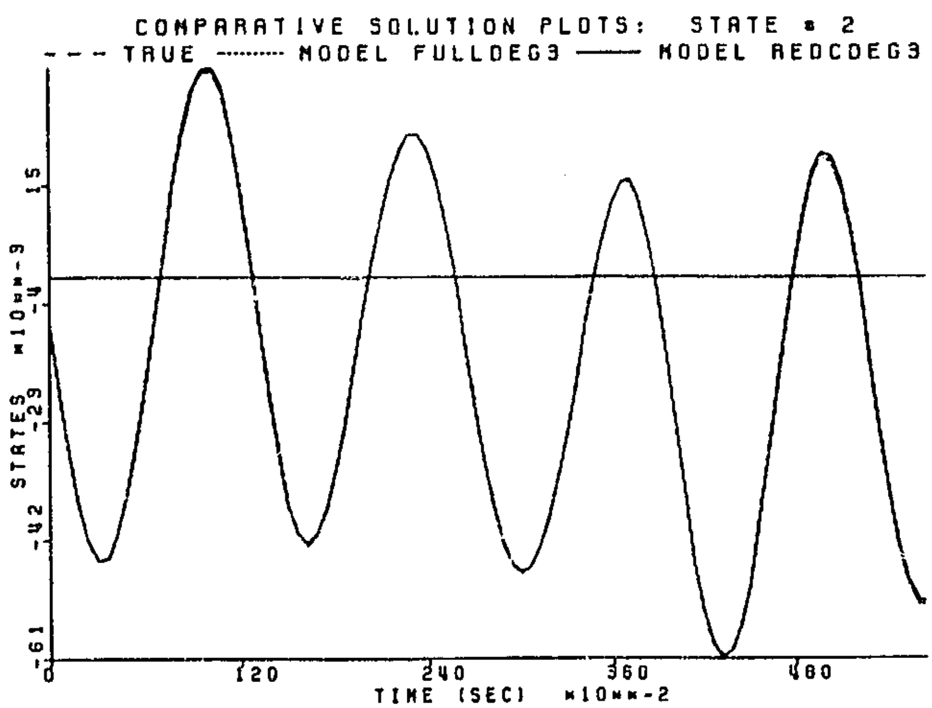
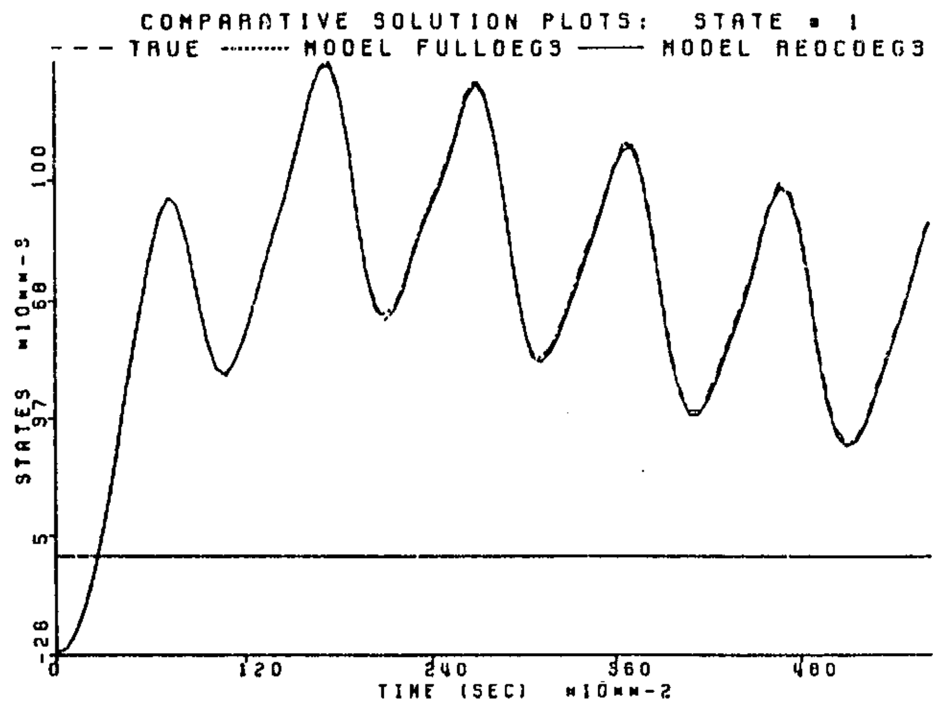




Figure 5.1.34 Simulation Number 7 of Table 5.1.22

```
**************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.116 | 0.110 |
| 2 | 0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.955E-01 | 0.938E-01 |
| 3 | 0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.395E-01 | 0.592E-01 |
| 4 | 0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.526E-01 | 0.726E-01 |
| 5 | 0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 3.75 | 5.24 |
| 6 | 0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 2.30 | 3.56 |
| 7 | 0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 7.59 | 15.9 |
| 8 | 0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 8.40 | 20.1 |
| 9 | -0.025 | 0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 0.187 | 0.211 |
| 10 | -0.025 | 0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 0.190 | 0.203 |
| 11 | -0.025 | 0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 0.205 | 0.277 |
| 12 | -0.025 | 0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 0.283 | 0.358 |
| 13 | -0.025 | -0.025 | 0.100 | 0.100 | 2.00 | 1.00 | 3.41 | 10.5 |
| 14 | -0.025 | -0.025 | 0.100 | -0.100 | 2.00 | 1.00 | 2.79 | 11.5 |
| 15 | -0.025 | -0.025 | -0.100 | 0.100 | 2.00 | 1.00 | 4.50 | 6.26 |
| 16 | -0.025 | -0.025 | -0.100 | -0.100 | 2.00 | 1.00 | 5.34 | 8.27 |
| 17 | 0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 1.51 | 3.20 |
| 18 | 0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 1.55 | 3.60 |
| 19 | 0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 7.07 | 6.06 |
| 20 | 0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 8.44 | 7.37 |
| 21 | 0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 14.3 | 4.16 |
| 22 | 0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 13.0 | 3.88 |
| 23 | 0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 8.03 | 3.34 |
| 24 | 0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 8.31 | 3.71 |
| 25 | -0.025 | 0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 2.99 | 3.89 |
| 26 | -0.025 | 0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 3.78 | 4.21 |
| 27 | -0.025 | 0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 8.96 | 4.57 |
| 28 | -0.025 | 0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 10.7 | 5.07 |
| 29 | -0.025 | -0.025 | 0.200 | 0.200 | 2.00 | 1.00 | 3.00 | 1.77 |
| 30 | -0.025 | -0.025 | 0.200 | -0.200 | 2.00 | 1.00 | 2.92 | 1.73 |
| 31 | -0.025 | -0.025 | -0.200 | 0.200 | 2.00 | 1.00 | 3.61 | 1.98 |
| 32 | -0.025 | -0.025 | -0.200 | -0.200 | 2.00 | 1.00 | 3.99 | 2.26 |
| 33 | 0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 5.33 | 1.34 |
| 34 | 0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 7.26 | 1.35 |
| 35 | 0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 9.18 | 1.70 |
| 36 | 0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 10.8 | 1.95 |
| 37 | 0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 4.30 | 0.970 |
| 38 | 0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 4.58 | 0.963 |
| 39 | 0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 4.54 | 1.24 |
| 40 | 0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 5.23 | 1.43 |
| 41 | -0.025 | 0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 4.19 | 1.07 |
| 42 | -0.025 | 0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 5.08 | 1.09 |
| 43 | -0.025 | 0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 6.03 | 1.41 |
| 44 | -0.025 | 0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 7.12 | 1.61 |
| 45 | -0.025 | -0.025 | 0.300 | 0.300 | 2.00 | 1.00 | 2.05 | 0.701 |
| 46 | -0.025 | -0.025 | 0.300 | -0.300 | 2.00 | 1.00 | 2.09 | 0.692 |
| 47 | -0.025 | -0.025 | -0.300 | 0.300 | 2.00 | 1.00 | 2.49 | 0.926 |
| 48 | -0.025 | -0.025 | -0.300 | -0.300 | 2.00 | 1.00 | 2.88 | 1.07 |

Table 5.1.23 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



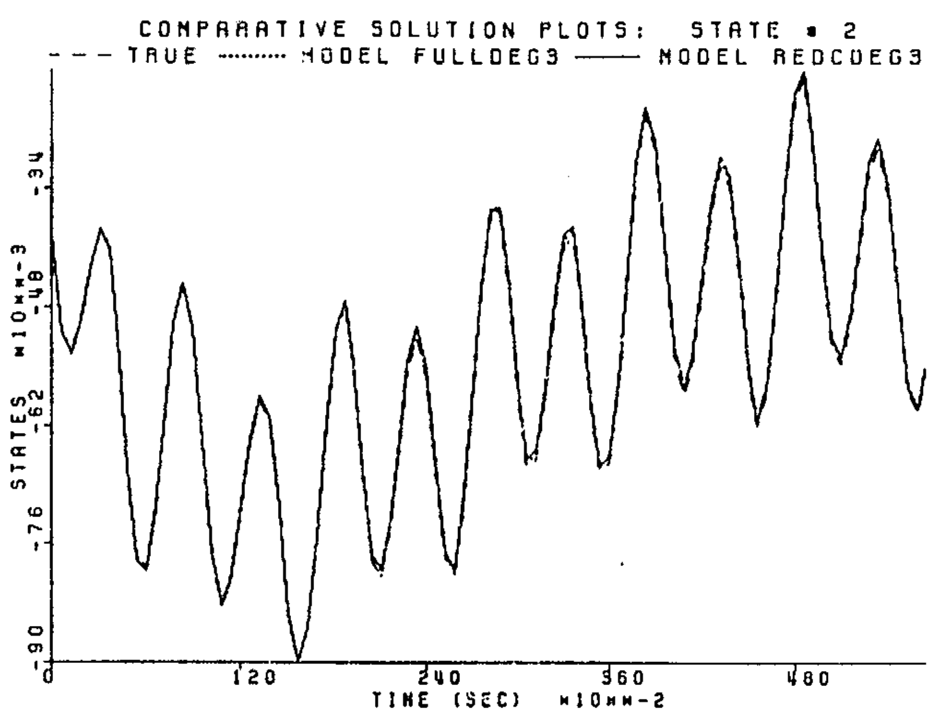
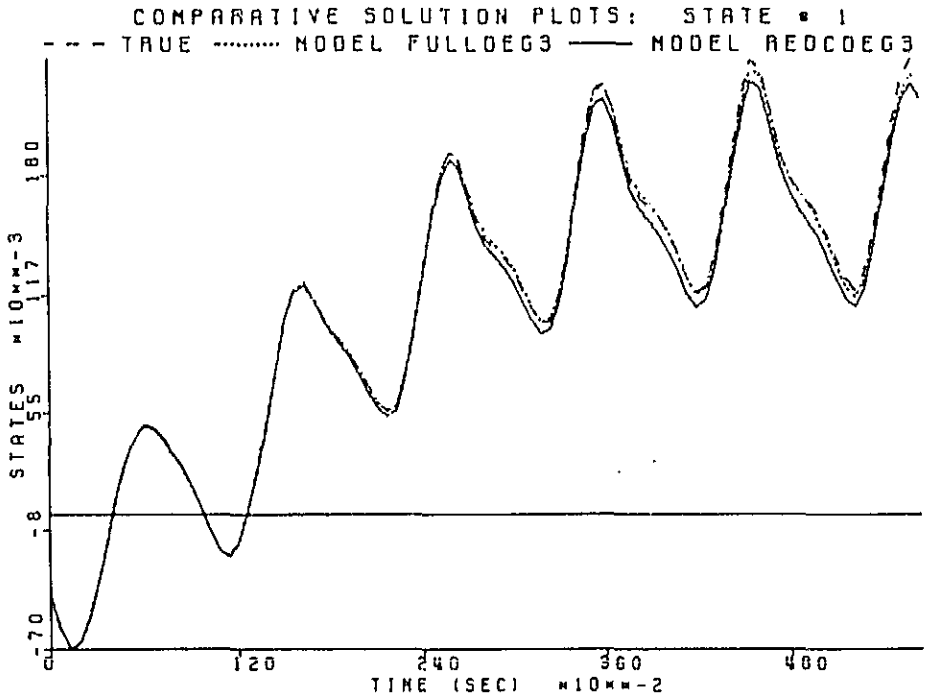




Figure 5.1.35 Simulation Number 30 of Table 5.1.23



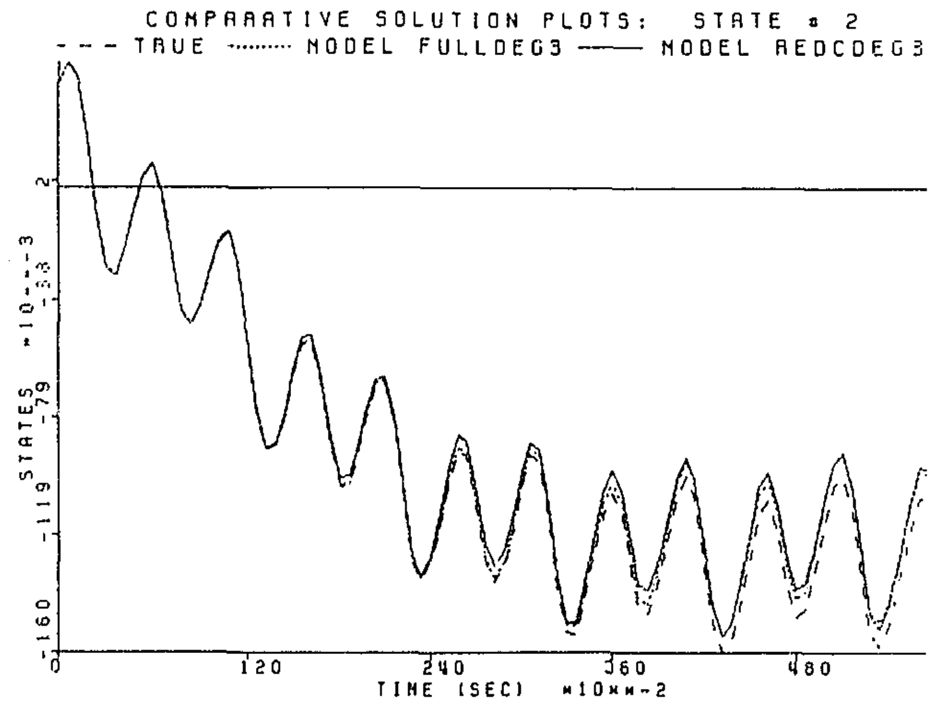
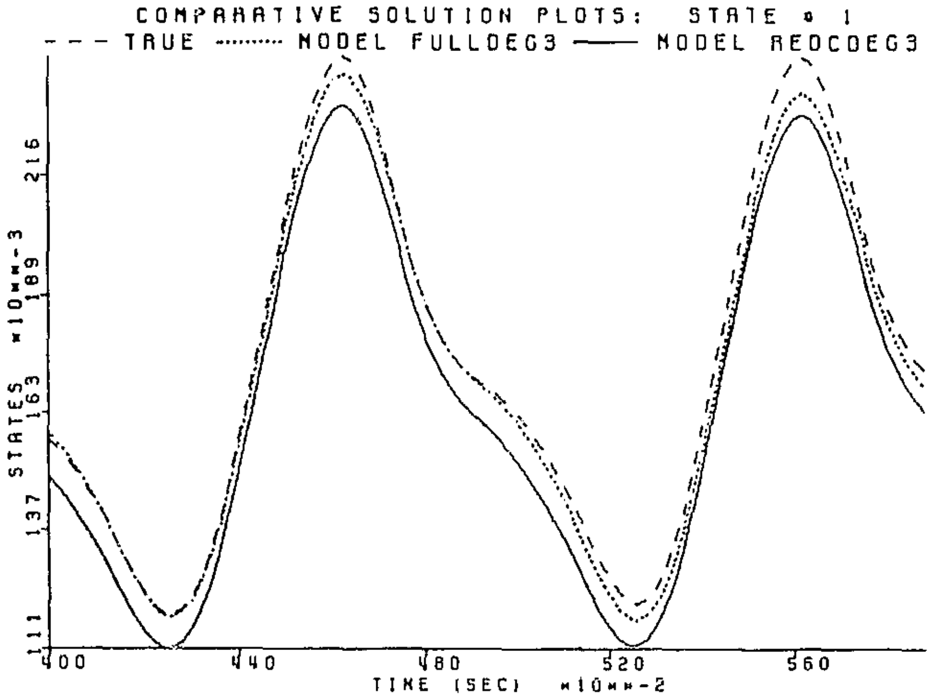




Figure 5.1.36 Simulation Number 44 of Table 5.1.23



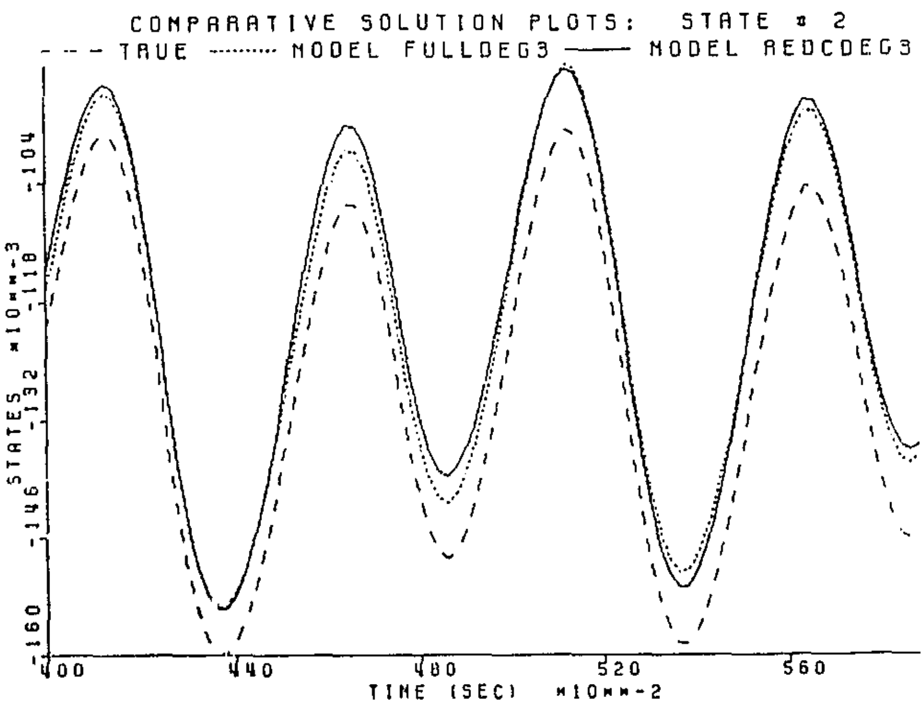




Figure 5.1.37 Expanded Section of Figure 5.1.36

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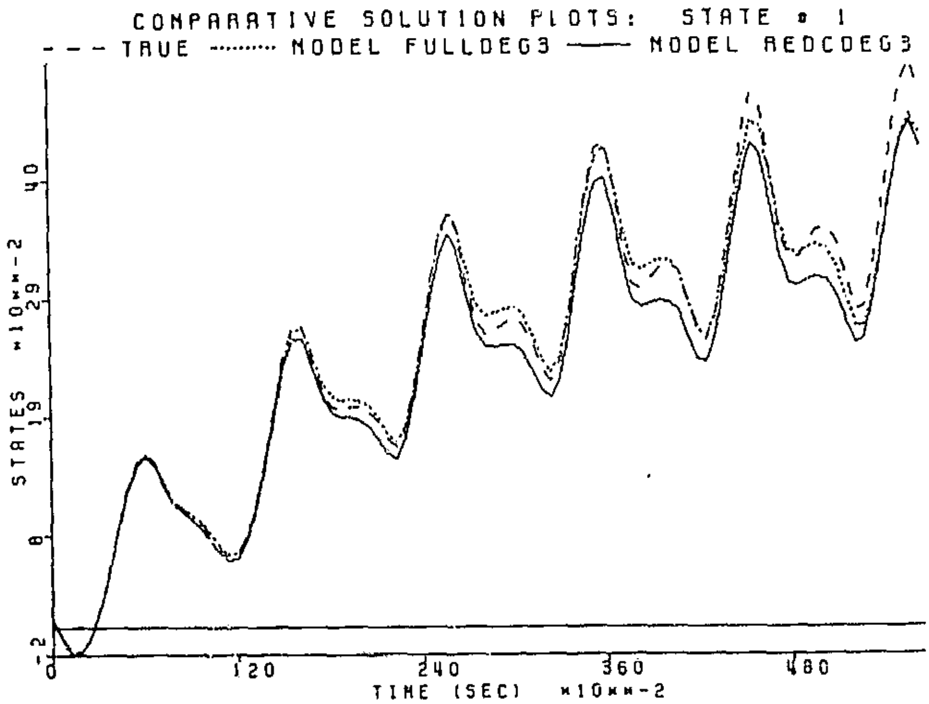
**************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.025 | 0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 2.12 | 0.478 |
| 2 | 0.025 | 0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 2.71 | 0.497 |
| 3 | 0.025 | 0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 4.32 | 0.698 |
| 4 | 0.025 | 0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 4.94 | 0.816 |
| 5 | 0.025 | -0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 1.84 | 0.395 |
| 6 | 0.025 | -0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 2.07 | 0.399 |
| 7 | 0.025 | -0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 2.57 | 0.555 |
| 8 | 0.025 | -0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 3.06 | 0.648 |
| 9 | -0.025 | 0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 1.82 | 0.417 |
| 10 | -0.025 | 0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 2.17 | 0.429 |
| 11 | -0.025 | 0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 3.17 | 0.606 |
| 12 | -0.025 | 0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 3.71 | 0.709 |
| 13 | -0.025 | -0.025 | 0.400 | 0.400 | 2.00 | 1.00 | 1.26 | 0.334 |
| 14 | -0.025 | -0.025 | 0.400 | -0.400 | 2.00 | 1.00 | 1.34 | 0.333 |
| 15 | -0.025 | -0.025 | -0.400 | 0.400 | 2.00 | 1.00 | 1.70 | 0.467 |
| 16 | -0.025 | -0.025 | -0.400 | -0.400 | 2.00 | 1.00 | 2.03 | 0.549 |
| 17 | 0.025 | 0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.307 | 0.110 |
| 18 | 0.025 | 0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.418 | 0.132 |
| 19 | 0.025 | 0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.919 | 0.231 |
| 20 | 0.025 | 0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.951 | 0.299 |
| 21 | 0.025 | -0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.318 | 0.801E-01 |
| 22 | 0.025 | -0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.410 | 0.920E-01 |
| 23 | 0.025 | -0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.723 | 0.165 |
| 24 | 0.025 | -0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.875 | 0.224 |
| 25 | -0.025 | 0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.288 | 0.894E-01 |
| 26 | -0.025 | 0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.380 | 0.106 |
| 27 | -0.025 | 0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.787 | 0.192 |
| 28 | -0.025 | 0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.875 | 0.255 |
| 29 | -0.025 | -0.025 | 0.500 | 0.500 | 2.00 | 1.00 | 0.254 | 0.596E-01 |
| 30 | -0.025 | -0.025 | 0.500 | -0.500 | 2.00 | 1.00 | 0.317 | 0.665E-01 |
| 31 | -0.025 | -0.025 | -0.500 | 0.500 | 2.00 | 1.00 | 0.518 | 0.127 |
| 32 | -0.025 | -0.025 | -0.500 | -0.500 | 2.00 | 1.00 | 0.678 | 0.181 |

Table 5.1.24 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



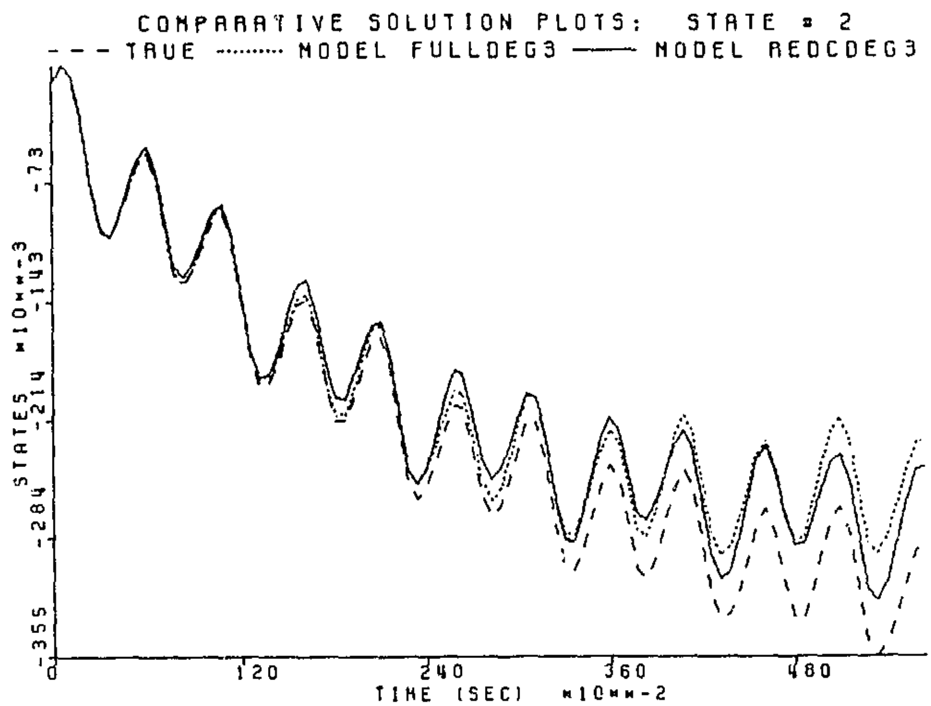




Figure 5.1.38 Simulation Number 8 of Table 5.1.24

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****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | -0.050 | -0.050 | -0.401 | 0.128 | 4.13 | 2.26 | 1.41 | 0.485 |
| 2 | -0.050 | -0.050 | 0.160 | 0.133 | 4.63 | 2.84 | 0.557 | 0.503 |
| 3 | -0.050 | -0.050 | 0.442 | -0.320 | 2.12 | 3.38 | 0.885 | 0.273 |
| 4 | -0.050 | -0.050 | -0.085 | -0.051 | 3.16 | 2.36 | 0.273E-01 | 0.420E-01 |
| 5 | -0.050 | -0.050 | 0.182 | 0.441 | 4.01 | 3.72 | 2.41 | 2.10 |
| 6 | -0.050 | -0.050 | 0.027 | 0.306 | 2.59 | 3.58 | 0.168E-01 | 0.195E-01 |
| 7 | -0.050 | -0.050 | 0.027 | 0.054 | 2.65 | 4.01 | 0.190 | 0.214 |
| 8 | -0.050 | -0.050 | 0.179 | 0.325 | 4.01 | 2.25 | 0.274 | 0.233 |
| 9 | -0.050 | -0.050 | -0.315 | -0.431 | 3.68 | 4.00 | 1.09 | 0.677 |
| 10 | -0.050 | -0.050 | 0.244 | 0.131 | 3.03 | 2.65 | 1.57 | 1.04 |
| 11 | -0.050 | -0.050 | 0.233 | 0.208 | 4.29 | 5.54 | 1.01 | 0.684 |
| 12 | -0.050 | -0.050 | -0.308 | 0.001 | 4.98 | 3.72 | 1.62 | 0.799 |
| 13 | -0.050 | -0.050 | -0.368 | 0.140 | 5.56 | 2.95 | 1.48 | 0.572 |
| 14 | -0.050 | -0.050 | -0.084 | -0.285 | 5.38 | 3.84 | 0.376E-02 | 0.143E-01 |
| 15 | -0.050 | -0.050 | 0.299 | -0.147 | 3.14 | 2.88 | 1.57 | 0.807 |
| 16 | -0.050 | -0.050 | 0.319 | -0.159 | 5.56 | 5.36 | 0.821 | 0.537 |
| 17 | -0.050 | -0.050 | -0.318 | 0.363 | 4.84 | 5.60 | 0.937 | 0.472 |
| 18 | -0.050 | -0.050 | -0.239 | -0.343 | 3.87 | 4.33 | 0.765 | 0.572 |
| 19 | -0.050 | -0.050 | -0.173 | 0.414 | 4.25 | 4.06 | 1.96 | 1.85 |
| 20 | -0.050 | -0.050 | -0.347 | -0.064 | 3.40 | 4.48 | 1.51 | 0.635 |
| 21 | -0.050 | -0.050 | 0.267 | 0.438 | 4.00 | 2.39 | 0.548 | 0.322 |
| 22 | -0.050 | -0.050 | 0.423 | 0.153 | 2.08 | 3.57 | 1.22 | 0.365 |
| 23 | -0.050 | -0.050 | 0.041 | 0.365 | 5.66 | 3.23 | 0.858E-02 | 0.103E-01 |
| 24 | -0.050 | -0.050 | -0.262 | 0.126 | 2.67 | 5.00 | 1.33 | 0.813 |
| 25 | -0.050 | -0.050 | -0.127 | -0.433 | 4.19 | 2.07 | 0.576E-01 | 0.790E-01 |
| 26 | -0.050 | -0.050 | -0.225 | 0.329 | 4.35 | 3.19 | 0.555 | 0.392 |
| 27 | -0.050 | -0.050 | 0.020 | -0.008 | 3.14 | 2.91 | 0.260 | 0.294 |
| 28 | -0.050 | -0.050 | 0.402 | 0.083 | 5.20 | 4.54 | 1.50 | 0.509 |
| 29 | -0.050 | -0.050 | 0.370 | -0.295 | 5.56 | 2.06 | 1.27 | 0.492 |
| 30 | -0.050 | -0.050 | -0.105 | -0.250 | 2.30 | 3.72 | 0.250E-01 | 0.414E-01 |
| 31 | -0.050 | -0.050 | 0.151 | -0.268 | 4.92 | 4.63 | 1.34 | 1.48 |
| 32 | -0.050 | -0.050 | -0.078 | -0.270 | 4.08 | 4.69 | 0.391E-01 | 0.887E-01 |
| 33 | -0.050 | -0.050 | 0.066 | -0.307 | 5.55 | 2.45 | 0.127E-02 | 0.506E-02 |
| 34 | -0.050 | -0.050 | 0.394 | 0.139 | 2.89 | 5.62 | 1.38 | 0.475 |
| 35 | -0.050 | -0.050 | 0.386 | -0.324 | 2.09 | 4.98 | 1.07 | 0.374 |
| 36 | -0.050 | -0.050 | 0.251 | -0.342 | 4.71 | 5.03 | 1.29 | 0.857 |
| 37 | -0.050 | -0.050 | 0.336 | 0.368 | 3.66 | 2.50 | 1.03 | 0.467 |
| 38 | -0.050 | -0.050 | -0.429 | -0.317 | 3.12 | 2.35 | 1.07 | 0.358 |
| 39 | -0.050 | -0.050 | -0.418 | 0.328 | 5.15 | 3.49 | 1.14 | 0.393 |
| 40 | -0.050 | -0.050 | 0.133 | -0.319 | 3.00 | 3.39 | 0.719 | 0.937 |
| 41 | -0.050 | -0.050 | -0.096 | 0.362 | 4.70 | 4.08 | 0.928E-01 | 0.168 |
| 42 | -0.050 | -0.050 | -0.156 | -0.370 | 2.46 | 4.10 | 0.870E-01 | 0.872E-01 |
| 43 | -0.050 | -0.050 | 0.334 | -0.431 | 3.13 | 4.73 | 0.800 | 0.367 |
| 44 | -0.050 | -0.050 | -0.202 | 0.151 | 2.74 | 3.98 | 0.884 | 0.696 |
| 45 | -0.050 | -0.050 | -0.449 | -0.062 | 2.25 | 2.36 | 1.45 | 0.385 |

Table 5.1.25 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model

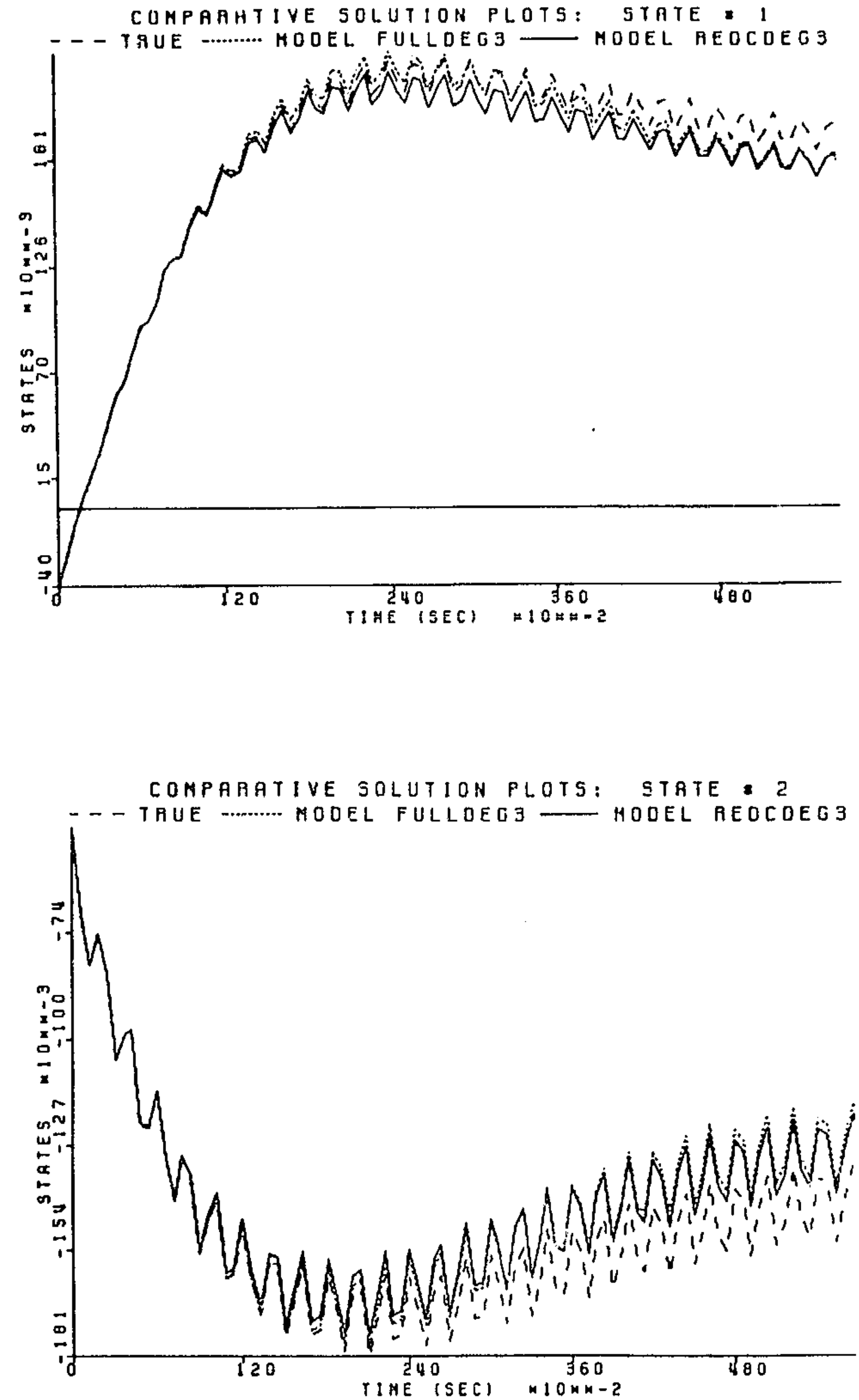
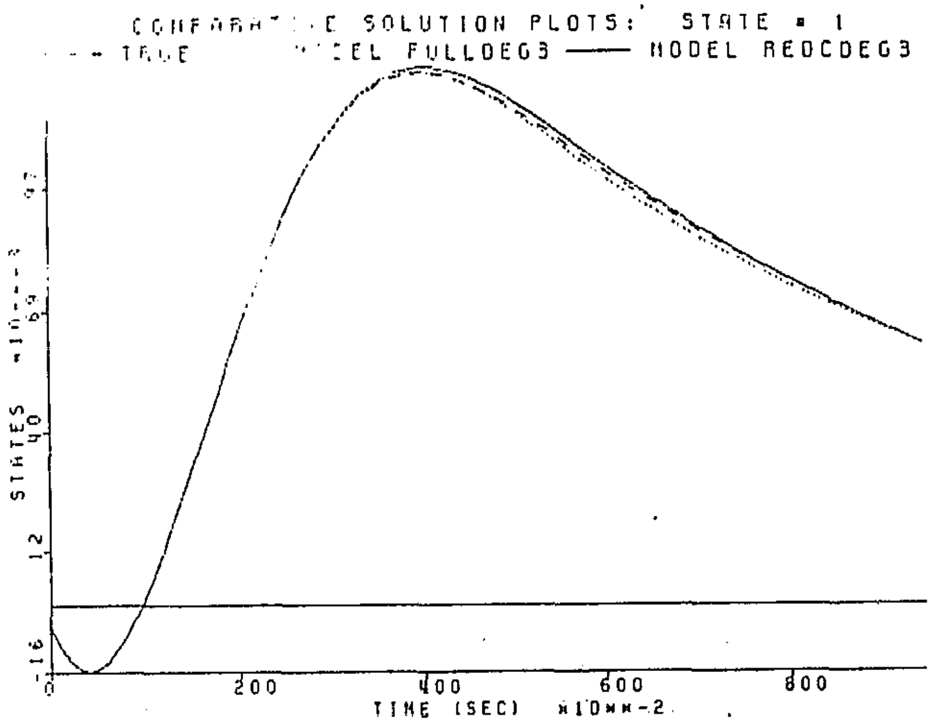



Figure 5.1.39 Simulation Number 3 of Table 5.1.25

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001 | -0.001 | 0.010 | -0.075 | 0.00 | 0.00 | 1.42 | 0.534 |
| 2 | 0.001 | -0.001 | 0.010 | -0.075 | 0.01 | 0.01 | 1.16 | 0.492 |
| 3 | 0.001 | -0.001 | 0.010 | -0.075 | 0.02 | 0.02 | 0.771 | 0.467 |
| 4 | 0.001 | -0.001 | 0.010 | -0.075 | 0.05 | 0.05 | 0.496 | 0.209 |
| 5 | 0.001 | -0.001 | -0.050 | 0.050 | 0.00 | 0.00 | 0.122E-01 | 0.311E-02 |
| 6 | 0.001 | -0.001 | -0.050 | 0.050 | 0.01 | 0.01 | 0.961E-02 | 0.274E-02 |
| 7 | 0.001 | -0.001 | -0.050 | 0.050 | 0.02 | 0.02 | 0.337E-02 | 0.117E-02 |
| 8 | 0.001 | -0.001 | -0.050 | 0.050 | 0.05 | 0.05 | 0.245E-02 | 0.229E-02 |
| 9 | 0.001 | -0.001 | -0.075 | 0.010 | 0.00 | 0.00 | 0.272E-01 | 0.671E-02 |
| 10 | 0.001 | -0.001 | -0.075 | 0.010 | 0.01 | 0.01 | 0.228E-01 | 0.627E-02 |
| 11 | 0.001 | -0.001 | -0.075 | 0.010 | 0.02 | 0.02 | 0.129E-01 | 0.433E-02 |
| 12 | 0.001 | -0.001 | -0.075 | 0.010 | 0.05 | 0.05 | 0.791E-02 | 0.473E-02 |
| 13 | 0.001 | -0.001 | -0.075 | -0.075 | 0.00 | 0.00 | 0.541E-01 | 0.335E-01 |
| 14 | 0.001 | -0.001 | -0.075 | -0.075 | 0.01 | 0.01 | 0.426E-01 | 0.267E-01 |
| 15 | 0.001 | -0.001 | -0.075 | -0.075 | 0.02 | 0.02 | 0.219E-01 | 0.164E-01 |
| 16 | 0.001 | -0.001 | -0.075 | -0.075 | 0.05 | 0.05 | 0.839E-02 | 0.656E-02 |
| 17 | 0.010 | -0.010 | 0.010 | -0.075 | 0.00 | 0.00 | 1.48 | 0.538 |
| 18 | 0.010 | -0.010 | 0.010 | -0.075 | 0.01 | 0.01 | 1.20 | 0.495 |
| 19 | 0.010 | -0.010 | 0.010 | -0.075 | 0.02 | 0.02 | 0.791 | 0.468 |
| 20 | 0.010 | -0.010 | 0.010 | -0.075 | 0.05 | 0.05 | 0.504 | 0.208 |
| 21 | 0.010 | -0.010 | -0.050 | 0.050 | 0.00 | 0.00 | 0.110E-01 | 0.293E-02 |
| 22 | 0.010 | -0.010 | -0.050 | 0.050 | 0.01 | 0.01 | 0.858E-02 | 0.256E-02 |
| 23 | 0.010 | -0.010 | -0.050 | 0.050 | 0.02 | 0.02 | 0.283E-02 | 0.102E-02 |
| 24 | 0.010 | -0.010 | -0.050 | 0.050 | 0.05 | 0.05 | 0.213E-02 | 0.215E-02 |
| 25 | 0.010 | -0.010 | -0.075 | 0.010 | 0.00 | 0.00 | 0.246E-01 | 0.629E-02 |
| 26 | 0.010 | -0.010 | -0.075 | 0.010 | 0.01 | 0.01 | 0.206E-01 | 0.584E-02 |
| 27 | 0.010 | -0.010 | -0.075 | 0.010 | 0.02 | 0.02 | 0.116E-01 | 0.392E-02 |
| 28 | 0.010 | -0.010 | -0.075 | 0.010 | 0.05 | 0.05 | 0.696E-02 | 0.430E-02 |
| 29 | 0.010 | -0.010 | -0.075 | -0.075 | 0.00 | 0.00 | 0.530E-01 | 0.333E-01 |
| 30 | 0.010 | -0.010 | -0.075 | -0.075 | 0.01 | 0.01 | 0.417E-01 | 0.264E-01 |
| 31 | 0.010 | -0.010 | -0.075 | -0.075 | 0.02 | 0.02 | 0.205E-01 | 0.155E-01 |
| 32 | 0.010 | -0.010 | -0.075 | -0.075 | 0.05 | 0.05 | 0.779E-02 | 0.599E-02 |

Table 5.1.26 Low Frequency Table for Third Degree Full Model versus Third Degree Reduced Model



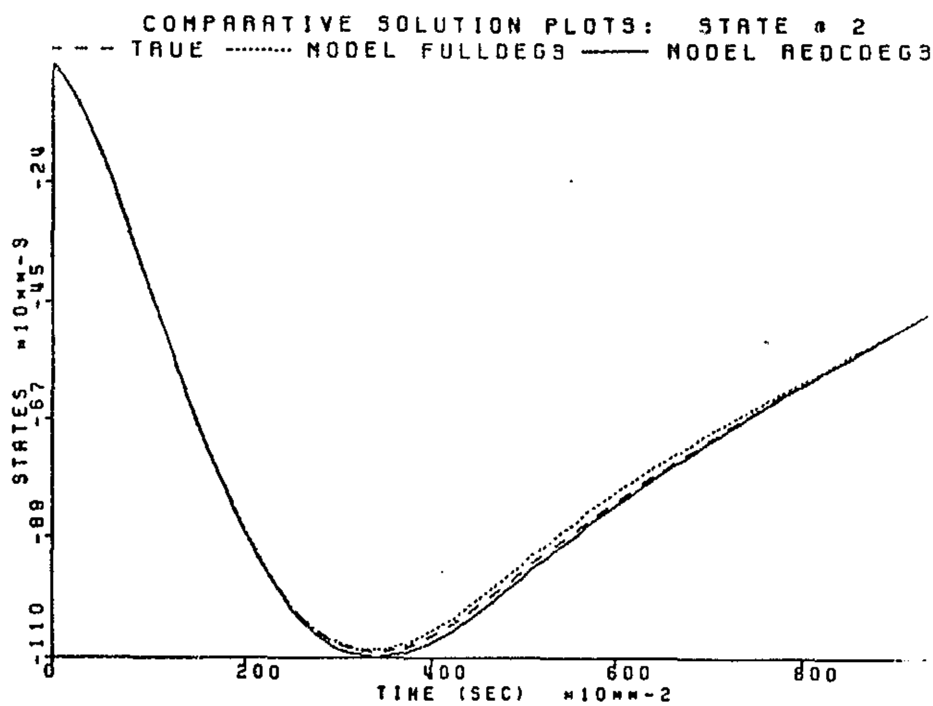




Figure 5.1.40 Simulation Number 3 of Table 5.1.26

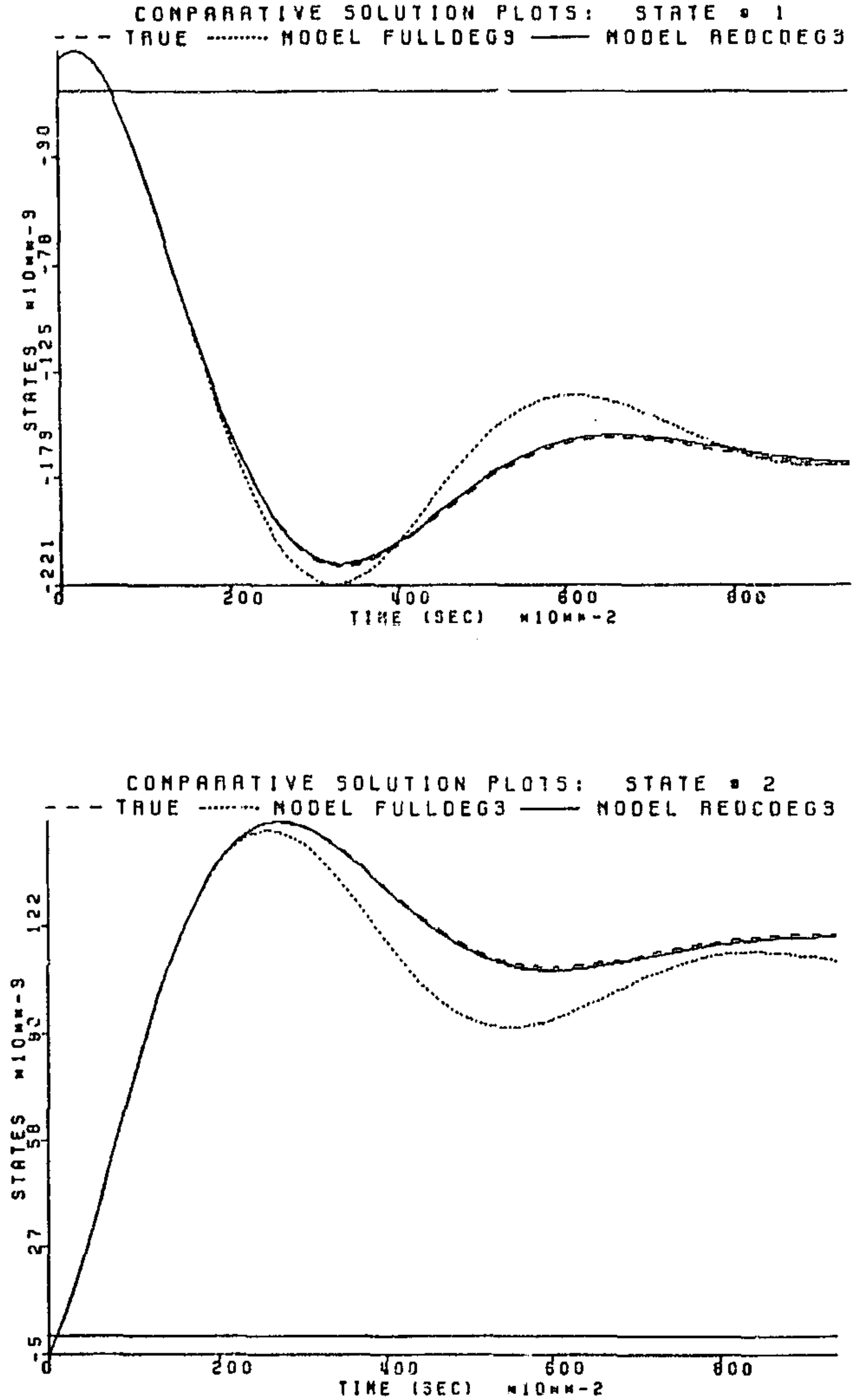
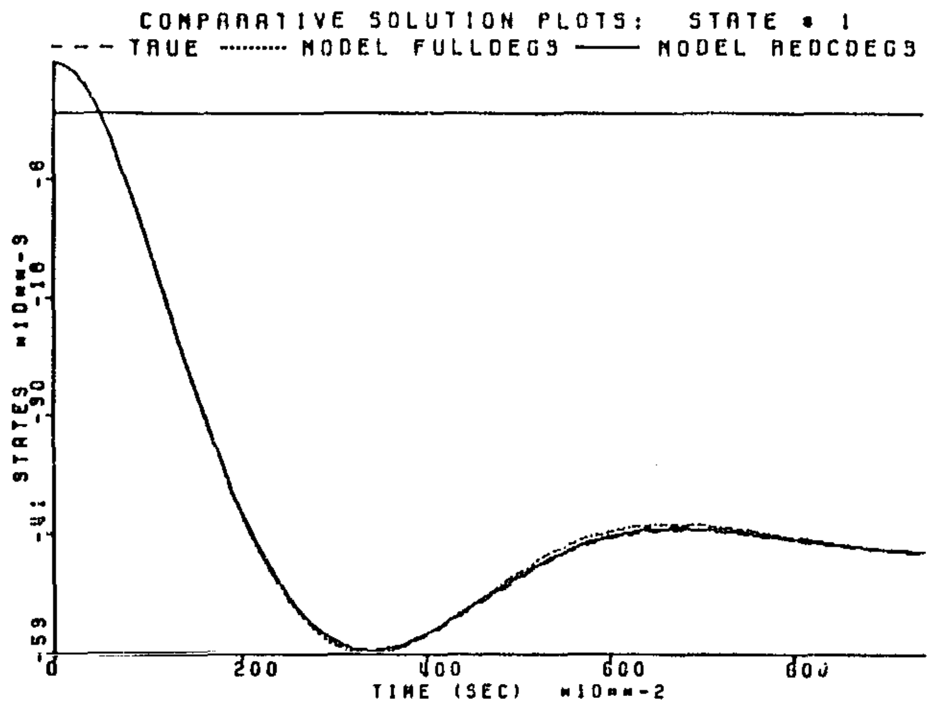



Figure 5.1.41 Simulation Number 21 of Table 5.1.26

```
**************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
**************************************
```

| S# | INITIAL CONDITIONS | | AMPLITUDES | | FREQUENCIES | | ERROR RATIOS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.000 | 0.010 | -0.022 | -0.008 | 0.00 | 0.00 | 0.105 | 0.383E-01 |
| 2 | -0.006 | 0.004 | 0.008 | -0.011 | 0.00 | 0.00 | 6.67 | 9.18 |
| 3 | 0.009 | 0.006 | -0.010 | 0.017 | 0.00 | 0.00 | 0.119 | 0.446E-01 |
| 4 | 0.006 | 0.010 | 0.022 | -0.010 | 0.00 | 0.00 | 0.136 | 0.644E-01 |
| 5 | 0.006 | -0.007 | -0.018 | -0.003 | 0.00 | 0.00 | 0.137 | 0.478E-01 |
| 6 | -0.004 | 0.001 | -0.021 | -0.005 | 0.00 | 0.00 | 0.144 | 0.457E-01 |
| 7 | 0.006 | 0.003 | 0.012 | 0.018 | 0.00 | 0.00 | 42.3 | 10.4 |
| 8 | -0.009 | -0.003 | -0.021 | 0.024 | 0.00 | 0.00 | 0.510E-01 | 0.179E-01 |
| 9 | -0.000 | 0.003 | 0.004 | 0.003 | 0.00 | 0.00 | 0.273 | 0.107 |
| 10 | -0.000 | 0.002 | -0.009 | -0.013 | 0.00 | 0.00 | 0.346 | 0.153 |
| 11 | 0.009 | 0.001 | 0.011 | 0.011 | 0.00 | 0.00 | 19.5 | 7.18 |
| 12 | 0.002 | -0.008 | 0.017 | 0.002 | 0.00 | 0.00 | 1.40 | 2.19 |
| 13 | 0.005 | -0.008 | -0.019 | -0.010 | 0.00 | 0.00 | 0.115 | 0.398E-01 |
| 14 | -0.004 | -0.005 | 0.022 | -0.018 | 0.00 | 0.00 | 0.288E-01 | 0.143E-01 |
| 15 | -0.007 | 0.007 | -0.001 | -0.020 | 0.00 | 0.00 | 8.16 | 4.11 |
| 16 | -0.010 | 0.005 | -0.005 | -0.006 | 0.00 | 0.00 | 0.740 | 0.245 |
| 17 | -0.007 | 0.006 | -0.018 | 0.009 | 0.00 | 0.00 | 0.137 | 0.439E-01 |
| 18 | -0.001 | -0.004 | 0.015 | 0.021 | 0.00 | 0.00 | 17.3 | 5.51 |
| 19 | 0.008 | -0.005 | 0.007 | 0.017 | 0.00 | 0.00 | 15.8 | 8.09 |
| 20 | 0.003 | -0.003 | -0.016 | -0.008 | 0.00 | 0.00 | 0.137 | 0.460E-01 |
| 21 | 0.004 | -0.003 | 0.019 | 0.006 | 0.00 | 0.00 | 0.553 | 1.47 |
| 22 | -0.003 | 0.006 | -0.010 | 0.007 | 0.00 | 0.00 | 0.227 | 0.749E-01 |
| 23 | 0.007 | -0.002 | 0.014 | -0.012 | 0.00 | 0.00 | 2.91 | 2.87 |
| 24 | -0.003 | -0.004 | 0.010 | -0.009 | 0.00 | 0.00 | 5.69 | 9.72 |
| 25 | -0.000 | -0.009 | -0.001 | -0.013 | 0.00 | 0.00 | 5.21 | 2.40 |
| 26 | -0.009 | 0.007 | 0.007 | -0.015 | 0.00 | 0.00 | 7.88 | 9.36 |
| 27 | -0.002 | 0.002 | -0.006 | -0.023 | 0.00 | 0.00 | 6.99 | 3.40 |
| 28 | 0.009 | 0.002 | -0.012 | 0.012 | 0.00 | 0.00 | 0.133 | 0.486E-01 |
| 29 | -0.006 | -0.007 | -0.003 | 0.006 | 0.00 | 0.00 | 0.592 | 0.231 |
| 30 | 0.004 | -0.000 | -0.000 | -0.009 | 0.00 | 0.00 | 5.78 | 2.41 |
| 31 | 0.001 | -0.007 | 0.022 | 0.014 | 0.00 | 0.00 | 9.40 | 7.24 |
| 32 | 0.008 | -0.002 | 0.004 | 0.024 | 0.00 | 0.00 | 1.27 | 0.539 |
| 33 | 0.001 | -0.007 | -0.023 | 0.018 | 0.00 | 0.00 | 0.633E-01 | 0.225E-01 |
| 34 | 0.003 | -0.005 | -0.024 | 0.024 | 0.00 | 0.00 | 0.464E-01 | 0.163E-01 |
| 35 | -0.001 | -0.007 | 0.017 | 0.012 | 0.00 | 0.00 | 5.05 | 1.29 |
| 36 | -0.010 | -0.006 | 0.015 | 0.014 | 0.00 | 0.00 | 7.37 | 2.39 |
| 37 | -0.009 | -0.006 | -0.004 | 0.002 | 0.00 | 0.00 | 0.429 | 0.167 |
| 38 | 0.001 | 0.008 | -0.005 | 0.020 | 0.00 | 0.00 | 0.201 | 0.740E-01 |
| 39 | 0.000 | -0.001 | -0.004 | 0.025 | 0.00 | 0.00 | 0.155 | 0.613E-01 |
| 40 | 0.003 | -0.009 | -0.011 | 0.009 | 0.00 | 0.00 | 0.174 | 0.647E-01 |
| 41 | -0.000 | 0.004 | 0.011 | -0.006 | 0.00 | 0.00 | 4.88 | 6.83 |
| 42 | 0.008 | -0.005 | -0.010 | 0.010 | 0.00 | 0.00 | 0.185 | 0.678E-01 |
| 43 | -0.009 | -0.008 | 0.012 | -0.021 | 0.00 | 0.00 | 0.824 | 0.566 |
| 44 | 0.003 | -0.007 | -0.021 | 0.015 | 0.00 | 0.00 | 0.780E-01 | 0.278E-01 |
| 45 | 0.001 | 0.007 | -0.016 | 0.012 | 0.00 | 0.00 | 0.117 | 0.401E-01 |
| 46 | -0.008 | 0.004 | 0.005 | 0.015 | 0.00 | 0.00 | 7.61 | 3.71 |
| 47 | -0.001 | -0.002 | 0.015 | 0.010 | 0.00 | 0.00 | 3.31 | 1.07 |
| 48 | -0.007 | 0.009 | 0.018 | -0.002 | 0.00 | 0.00 | 1.11 | 0.993 |
| 49 | 0.006 | -0.005 | 0.004 | 0.008 | 0.00 | 0.00 | 39.4 | 27.1 |
| 50 | -0.006 | 0.003 | -0.011 | -0.021 | 0.00 | 0.00 | 2.60 | 1.68 |

Table 5.1.27 Step Response Table for Third Degree Full Model versus Third Degree Reduced Model



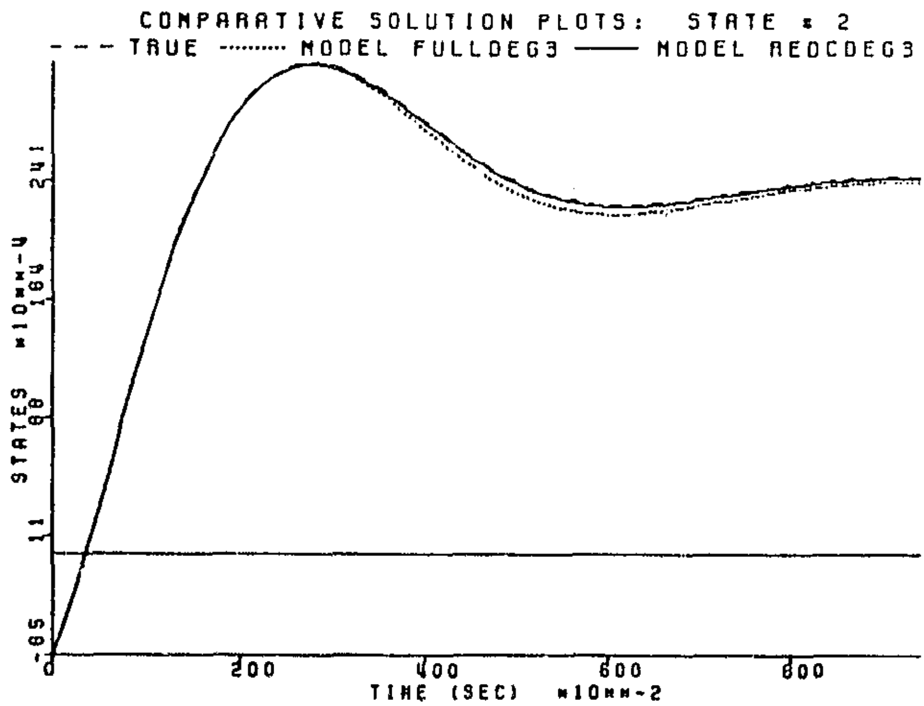
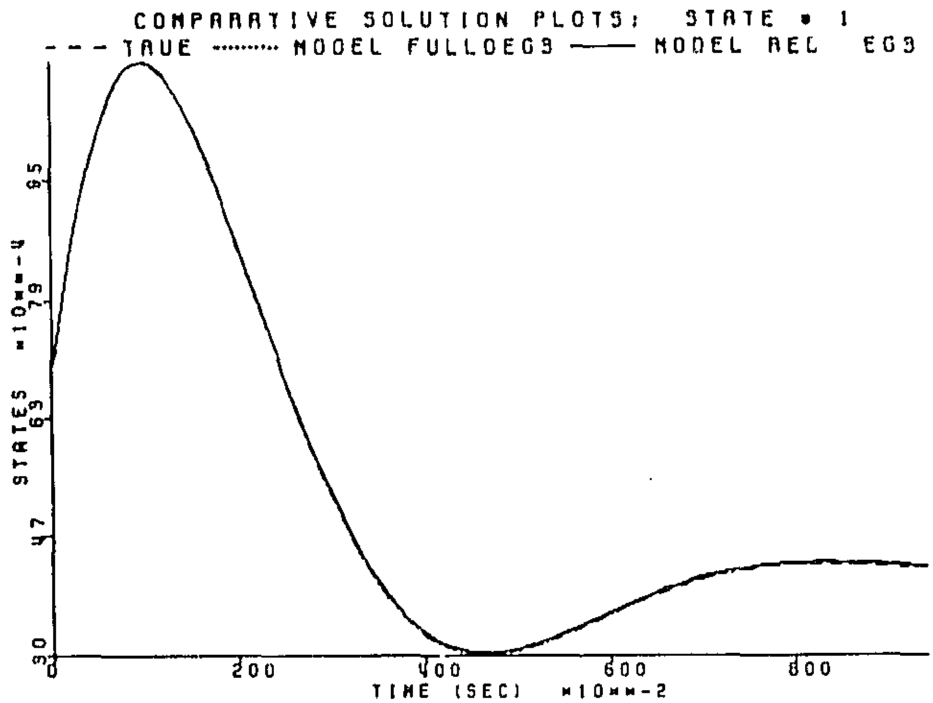




Figure 5.1.42 Simulation Number 13 of Table 5.1.27



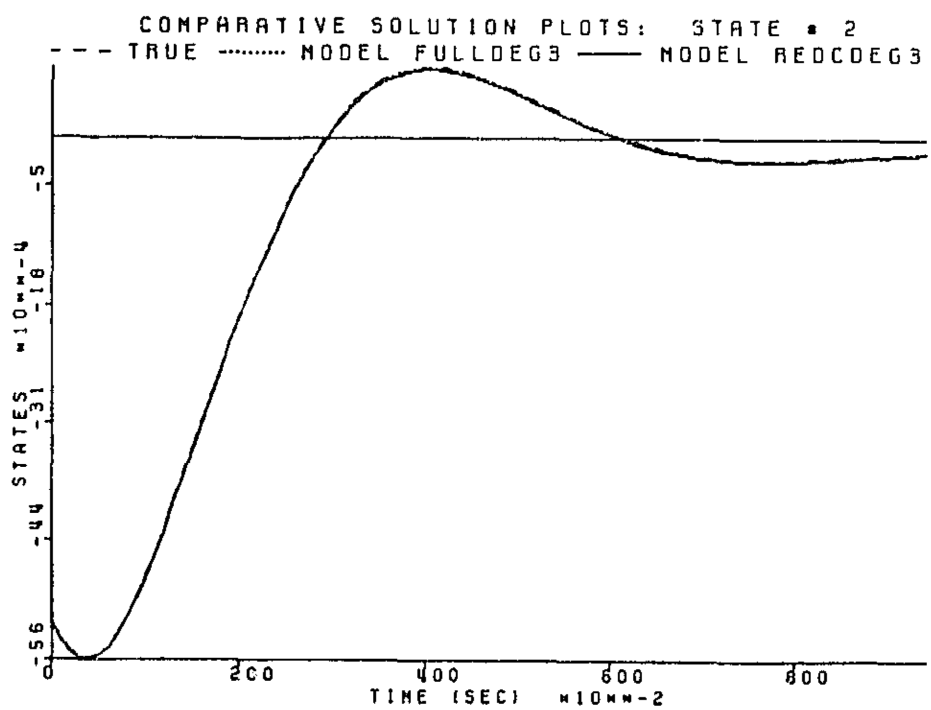




Figure 5.1.43 Simulation Number 49 of Table 5.1.27

duced model behavior is actually better than the full model behavior. So a compromise between the full and reduced models is met, the full model is stable for bigger amplitudes (that is, a larger region of stability), but the reduced model has better low frequency behavior. It is not the purpose here to resolve this decision, but rather to demonstrate that a significantly reduced model can approximate the full model without much loss of higher degree dynamics. Indeed, this was shown.

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